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TECHNICAL COMMUNICATION

A Single Equation for Calculating z_{0N} for Hydraulically Smooth, Transitionally Rough, and Fully Rough Turbulent Flows

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ABSTRACT

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The velocity-profile data from NIKURADSE's (1950) seminal study was reexamined with modern curve-fitting software. A single equation was derived that describes the influence of bottom roughness on the velocity profile over a wide range of flow conditions, including hydraulically smooth, transitionally rough, and fully rough turbulent flows.

ADDITIONAL INDEX WORDS: *Nikuradse's roughness parameter, sand-grain roughness, shear velocity, shear stress, roughness Reynolds number.*

In his seminal study on the influence of sand-grain roughness on the logarithmic velocity profiles of turbulent flows in pipes, NIKURADSE (1950) found good agreement with an expression of the form

$$U(z)/u_* = 5.75 \log(z/k_s) + A \quad (1)$$

where $U(z)$ is the flow speed at height z above the seabed, u_* is the shear velocity ($=\sqrt{\tau_b/\rho}$), τ_b is the shear stress at the seabed, ρ is fluid density, k_s is the sand-grain diameter, A is an empirically derived function of the roughness Reynolds number R_* ($= u_* k_s/v$), and v is the kinematic viscosity. Although (1) is seen occasionally in the engineering literature, it is written more commonly in terms of the natural logarithm,

$$\frac{U(z)}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z}{k_s}\right) + A \quad (2)$$

where K (= 0.41) is von Kármán's constant. Oceanographers generally simplify (2) further by redefining A as

$$A \equiv 1/\kappa \ln(k_s/z_{0N}) \quad (3)$$

to obtain

$$\frac{U(z)}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z}{z_{0N}}\right) \quad (4)$$

where the influence of A on the logarithmic velocity profile over sand-grain roughness has been incorporated into the Nikuradse roughness parameter, z_{0N}.

For hydraulically smooth turbulent flows and for fully rough turbulent flows, z_{0N} is described by relatively simple equations (Table 1). For transitionally rough turbulent flows, however, both A and z_{0N} are rapidly changing functions of R_*. They are calculated either by piecemeal regressions on segments of Nikuradse's data (*e.g.,* Table 1) or through graphical approximation (*e.g.,* SMITH, 1977, his Figure 1). In hopes of simplifying the calculation of A and z_{0N}, particularly for transitionally rough turbulent flows, I reevaluated Nikuradse's original data with modern curve-fitting software (TableCurve 2D for Windows 3.1, version 2.03, Jandel Scientific).

I tested the goodness of fit for a variety of functions with least-squares regressions on 403 data values (Figure 1) that I extracted from NIKURADSE's (1950) results. I found a suitable fit for a nonlinear, chemical-kinetics equation (with equilibrium) of the form

$$A = \mathrm{a} + \mathrm{b}\left\{1 - \exp(-\mathrm{c}R_*) - \frac{\mathrm{d}}{n}\left[1 + \frac{\mathrm{c}\exp(-nR_*) - n\exp(-\mathrm{c}R_*)}{n - \mathrm{c}}\right]\right\}$$
$$n = \mathrm{d} + \mathrm{e} \quad (5)$$

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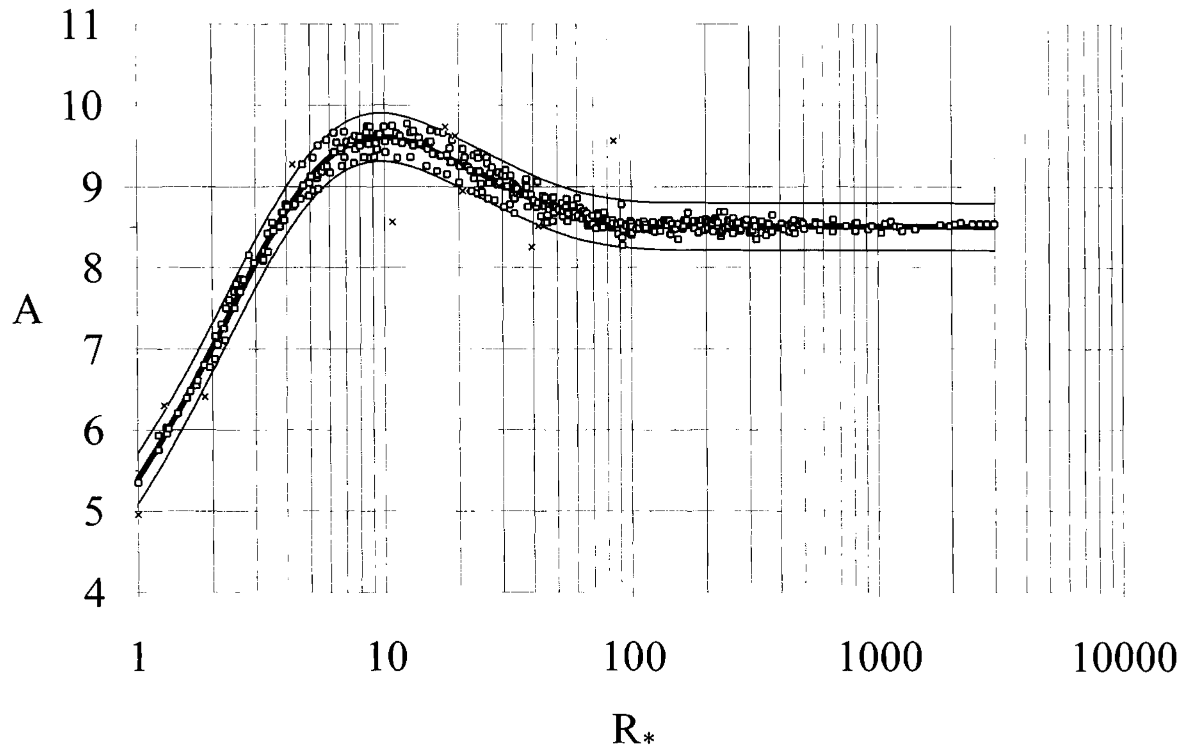

Figure 1. Nonlinear least-squares regression of NIKURADSE's (1950) original data with a chemical-kinetics equation (with equilibrium). The regression equation is shown as a thick line beneath the active data points (open squares). The 99% prediction intervals are shown as two thin lines bounding the data. Eleven outliers (×) that exceeded the 99% prediction intervals were omitted during the regression.

Table 1. *Equations for calculating* A *and* z_{ON} *for turbulent flows over grain-size roughness. Equations for* A *are from NIKURADSE (1950, his equations 21 a–e).* R_* *intervals and equations for* z_{ON} *are from WEATHERLY (1972, his equations 5–6, 13–15).*

	R_*	A	z_{0N}
Hydraulically smooth turbulent flow	$0 \leq R_* < 3$	$A = 5.5 + 5.75 \log R_*$	$z_{0N} = 0.1v/u_*$
transitionally rough turbulent flow	$3 \leq R_* \leq 7$	$A = 6.59 + 3.5 \log R_*$	$z_{0N} = \sqrt{k_s v/300u_*}$
	$7 \leq R_* \leq 14$	$A = 9.58$	$z_{0N} = k_s/44.4$
	$14 \leq R_* \leq 90$	$A = 11.5 - 1.62 \log R_*$	$z_{0N} = [u_* (k_s/30)^5/v]^{0.25}/1.21$
fully rough turbulent flow	$R_* > 90$	$A = 8.48$	$z_{0N} = k_s/30$

Table 2. *Regression coefficients and statistical measures for equation (5).*

	r^2 Coef Det	DF Adj r^2	Fit Std Err	F-value	
	0.9730	0.9726	0.1102	3481	
Coefficients	Value	Std Error	t-value	99% Confidence Limits	
a	2.905	0.1160	25.03	2.605	3.205
b	73.39	4.819	15.23	60.92	85.86
c	0.0420	0.001730	24.28	0.03754	0.04649
d	0.3927	0.01094	35.91	0.3644	0.4210
e	0.03245	0.001159	28.01	0.02945	0.03545

(eqn. 8133, TableCurve 2D) where a, b, c, d, and e are regression coefficients. To improve the goodness of fit, I removed eleven outliers that exceeded the 99% prediction intervals, and I ran another least-squares regression on the remaining data to obtain final values for the five regression coefficients and for the statistical measures of the regression (Table 2).

The resulting regression (Figure 1) provided an excellent fit over the entire range of Nikuradse's data, which included hydraulically smooth, transitionally rough, and fully rough turbulent flows. For fully rough turbulent flows (*i.e.*, for large values of R_*), (5) approaches an equilibrium value of 8.51, which is in excellent agreement with Nikuradse's value of 8.48. Given a value for A, one can calculate z_{0N} from

$$z_{0N} = k_s \exp(-\kappa A). \tag{6}$$

For fully rough turbulent flows, (6) approaches an equilibrium value of $k_s/33$, which is in good agreement with the traditional value of $k_s/30$.

Beyond the obvious convenience of having a single equation for calculating A and z_{0N} over a wide range of flow conditions, (5) has distinct advantages over piecemeal regressions in that the equation and its first derivative are continuous, smoothly varying functions. Thus, to obtain an estimate of A or z_{0N} from a single measurement of current speed in the logarithmic layer, one can simply construct a Newton-Raphson iteration on R_* of the form

$$R_*^{(n+1)} = R_*^{(n)} - \frac{\dfrac{1}{\kappa}\ln\left(\dfrac{z_r}{k_s}\right) - \dfrac{k_s U_r}{\nu R_*^{(n)}} + A^{(n)}}{\dfrac{k_s U_r}{\nu [R_*^{(n)}]^2} + \mathrm{bc}\left\{\exp(-\mathrm{c}R_*^{(n)}) + \mathrm{d}\left[\dfrac{\exp(-nR_*^{(n)}) - \exp(-\mathrm{c}R_*^{(n)})}{n - \mathrm{c}}\right]\right\}} \tag{7}$$

where (n) and $(n + 1)$ denote sequential steps in the iteration process, $A^{(n)}$ is calculated from (5) using the value for $R_*^{(n)}$, and U_r is a current speed measured within the logarithmic layer at reference height z_r. Once the value for R_* has converged to the desired precision, A and z_{0N} can be calculated from (5) and (6), respectively, and $u_* = R_* \nu / k_s$. This iteration process may look complicated, but it can be easily programmed. It is particularly useful in modeling studies for determining the bottom boundary condition of hydraulically smooth, transitionally rough, or fully rough turbulent flows when A or z_{0N} vary spatially or temporally.

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