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Hyperbolic Distributed Wind, Sea-Level and Wave Data

Poul Skyum†, Christian Christiansen† and Preben Blæsild‡

†Department of Earth Sciences
University of Aarhus
Ny Munkegade, Building 520
DK-8000 Aarhus C, Denmark

‡Department of Theoretical Statistics
University of Aarhus
Ny Munkegade, Building 521
DK-8000 Aarhus C, Denmark

ABSTRACT

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The hyperbolic distribution has four parameters and the logarithm of its probability density function is a hyperbola. The distribution has been used to analyze data from different scientific areas and in particular data from earth science. Some of the most important properties of this flexible distribution are discussed. Good agreements are found when fitting the distribution to wind, sea-level and wave observations. These agreements are better than can be obtained when applying the traditionally used distributions such as the Weibull, the log-normal, and the Rayleigh distribution. Return periods calculated from the distribution are also in agreement with observations. A case of fitting the two-dimensional version of the distribution to a set of data consisting of simultaneous recordings of wave height and wave period is discussed.

ADDITIONAL INDEX WORDS: *Hyperbolic distribution, wind, sea-level, wave height, wave period.*

INTRODUCTION

Frequency analysis of wind, sea-level and wave data is an important element in the design of dikes and off-shore constructions and in the planning and management of coastal protections. It is often concerned with the search of a statistical distribution which could be used to fit adequately a given set of data. From the chosen distribution, one can make extrapolations in order to estimate extreme events corresponding to a high return period (100-year wave or sea-level for example). This estimation can then be used to design control structures that are able to withstand such extreme events.

Obviously, such extrapolations (predictions) are only of value if there is a good agreement between the observations and the fitted statistical distribution. The question of which distribution should be used to reach this objective has been discussed by several authors. Gaussian, log-normal, Rayleigh and Weibull distributions have often been used when dealing with sea-level and wave data (HOLMB, 1981). However, statistical features of waves are often not in agreement with such distributional models (TANG, 1986; GODA, 1988; GODA and KOBUNE, 1990). Therefore, some authors recommend the use of mixtures of statistical distributions. ROSEN and KIT (1981) in their study of waves used the log-normal distribution for the central part of their wave observations and the Weibull distribution for the extreme tails.

It is well known that wave-induced forces on designed structures arise from pressure, velocities and accelerations, all of which are proportional to wave height and depend on the wave period. There is therefore a considerable engineering interest in the joint distribution of wave heights and periods (NOLTE, 1979; LOSADA and GIMÉNEZ-CURTO, 1979). Present work on joint distribution of wave heights and periods (*e.g.*, DOERING and DONELAN, 1993) builds to a high degree on the formulations of LONGUET-HIGGINS (1975, 1983) where assumptions of gaussianity are inherent.

The 4-parameter hyperbolic distribution which will be the subject of our investigation has two shape parameters, one scale parameter and one location parameter. The hyperbolic distribution, therefore, provides more flexibility than the above 2 and 3 parameter distributions (CHRISTIANSEN and HARTMANN, 1991). Inspired by BAGNOLD's (1941) plots of log grain-size vs. log probability density, the hyperbolic distribution was introduced to describe the mass-size distribution of sand samples (BARNDORFF-NIELSEN, 1977). The distribution has wide application, not only to other kinds of sediments (BAGNOLD and BARNDORFF-NIELSEN, 1980; CHRISTIANSEN, 1984; CHRISTIANSEN *et al.*, 1984; BARNDORFF-NIELSEN and CHRISTIANSEN, 1988; CHRISTIANSEN and KRISTENSEN, 1988), but also to other types of frequency data: size distribution of oil fields (SEYEDGHASEMIPOUR and BHATTACHARYYA, 1990), size distributions of droplets and aerosols (DURST and MACAGNO, 1986), turbulence (BARNDORFF-NIELSEN, 1979), wind shear (BARNDORFF-NIELSEN *et al.*, 1989), and in astronomy, biology, and economics (BARNDORFF-NIELSEN and BLÆSILD, 1981, 1983; BARNDORFF-NIELSEN *et al.*, 1985). Data used in the present study come from the inner Danish waters (Figure 1).

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THE HYPERBOLIC DISTRIBUTION

The hyperbolic distribution is defined by its log probability function being a hyperbola, just as that of the normal distri-

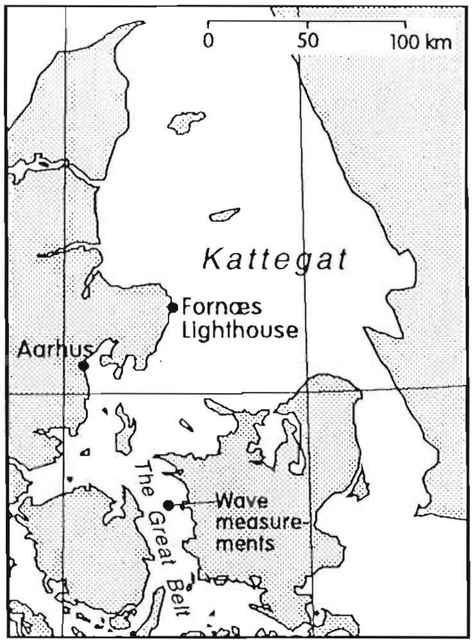

Figure 1. Map of the central part of the inner Danish waters, showing the sampling sites.

bution is a parabola. The hyperbolic distribution needs four parameters for its specification, two of which define the position and scale of the hyperbola and two which define the "shape" of the hyperbola. We give here one of the parametrizations of the model function:

$$p(x, \mu, \delta, \phi, \gamma) = a(\delta, \phi, \gamma)\exp[-\tfrac{1}{2}(\phi h_{-} + \gamma h_{+})], \quad (1)$$

where x indicates the observed variable; μ, δ, ϕ, and γ are parameters,

$$h_{\pm} = \sqrt{\delta^2 + (x - \mu)^2} \pm (x - \mu)$$

and

$$a(\delta, \phi, \gamma) = \frac{\sqrt{\phi\gamma}}{\delta(\phi + \gamma)K_1(\delta\sqrt{\phi\gamma})},$$

K_1 being a Bessel function. For fixed values of μ, δ, ϕ, and γ, equation (1) determines a probability (density) function.

Figure 2 shows the geometrical interpretations of the parameters and some of their useful combinations. The parameters ϕ and γ are simply the slopes of the two linear asymptotes of the hyperbolic log probability function. They therefore correspond to BAGNOLD's (1941) "small grade" and "coarse grade" coefficient, respectively.

Similarly, μ corresponds to the log of BAGNOLD's "peak diameter". Applied to sediments, we prefer not to use μ but

$$\nu = \mu + \frac{\delta(\phi - \gamma)}{2\sqrt{\phi\gamma}},$$

which is the mode point of the distribution and is referred to

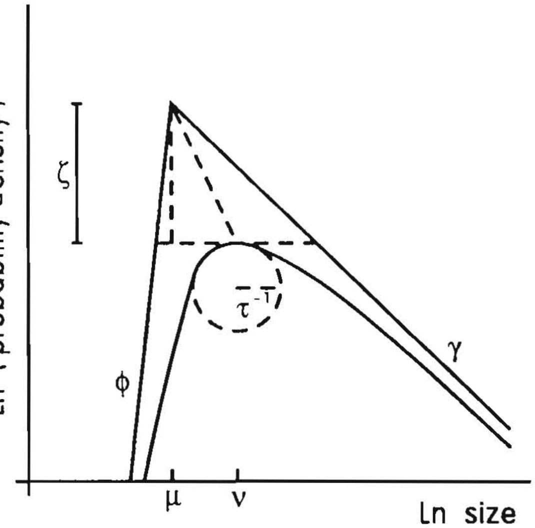

Figure 2. Geometrical interpretation of the main parameters in the hyperbolic distribution.

as the "typical log-grain size". The scale parameter δ has no direct interpretation in Figure 2; but

$$\zeta = \delta\sqrt{\phi\gamma}$$

is the difference between the ordinate of the log-hyperbolic curve at the mode point $x = \upsilon$ and the ordinate at the intersection point $x = \mu$ of the asymptotes.

The spread (sorting) of the distribution can be measured in different ways. Near the mode point it may be described by

$$\tau^2 = \delta^{-2}\zeta(1 - \rho^2)$$

where

$$\rho = \frac{\phi - \gamma}{\phi + \gamma}.$$

The parameter τ^2 represents the curvature of the hyperbola at the mode point $x = \nu$. The parameters δ, ζ, and $\kappa = (\phi\gamma)^{1/2}$ are also measures of spread (see Figure 2).

The skewness and the kurtosis of the log-hyperbolic distribution, as traditionally defined in statistics, are very complicated functions of ϕ, γ, and δ (BARNDORFF-NIELSEN and BLÆSILD, 1981). In most cases ($\zeta > 1$ and $|\rho| < 5^{-1/2} \approx 0.447$) we may approximate the kurtosis by

$$\xi = \frac{1}{\sqrt{1 + \zeta}} \quad (2)$$

and the skewness by

$$\chi = \rho\xi. \quad (3)$$

The domain of variation of χ and ξ, *i.e.*, $\{(\chi, \xi): 0 < |\chi| < \xi < 1\}$ is a triangle referred to as the hyperbolic shape triangle (Figure 3).

From the probability density function of the hyperbolic distribution (1) it may, for example, be seen that for $\xi = 0$ and

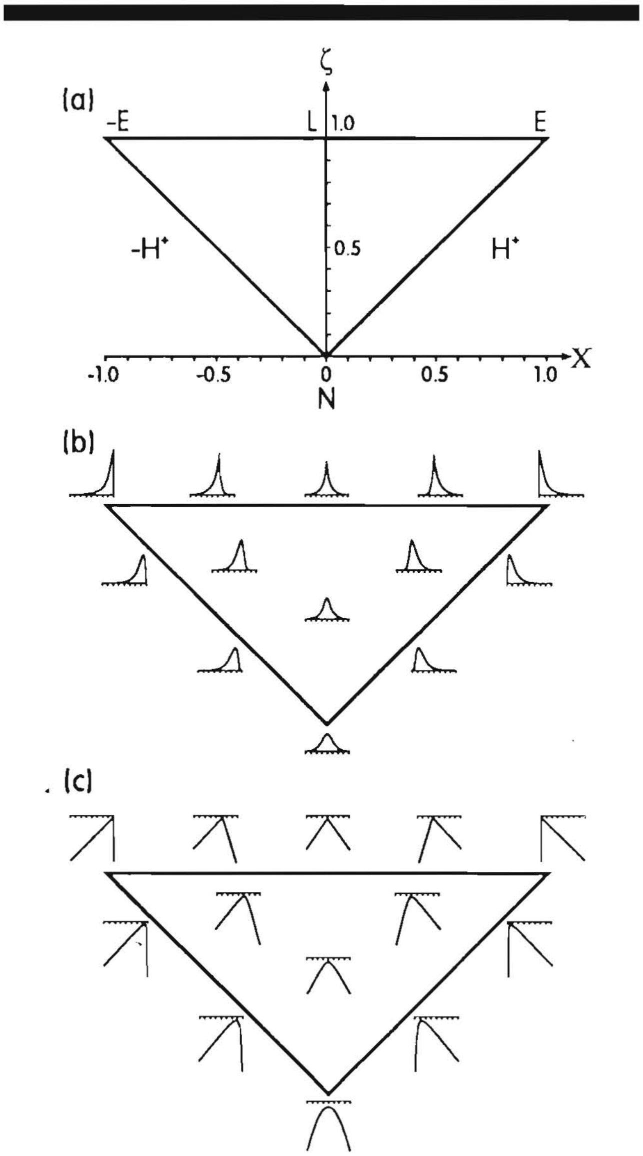

Figure 3. (a) The hyperbolic shape triangle, *i.e.*, the domain of variation of the invariant parameters χ and ξ of the hyperbolic distribution. The letters at the boundaries indicate how the normal distribution (N), the positive and negative hyperbolic distributions (H^+ and $-H^+$), the Laplace distribution (symmetrical or skew) (L), and the exponential distribution (E) are limits of the hyperbolic distribution. (b) Representative probability functions corresponding to selected (χ, ξ) values, including limiting forms of the hyperbolic distribution. The distributions have been selected so as to have variance equal to unity. (c) The logarithmic probability functions corresponding to (b).

$\chi = 0$ (see also (2) and (3)) we obtain the normal distribution, that for $\xi = 1$ and $-1 < \chi < 1$ we obtain the symmetrical and skew Laplace distributions, and that for $\xi = 1$ and $\chi = 1$ we obtain the exponential distributions as three of the limiting distributions of the hyperbolic distribution. Negatively skewed distributions plot to the left of the central line in the triangle and positively skewed distributions to the right in the triangle; the more peaked a distribution is the higher it plots in the triangle.

The hyperbolic distribution (1) belongs to a very large class of distribution called the generalized hyperbolic distributions, which were introduced by BARNDORFF-NIELSEN (1977) and studied further by BLÆSILD (1981). It is important to notice that these distributions may be defined as mixtures of normal distributions in the following way:

$$H_d(\lambda, \alpha, \beta, \delta, \mu, \Delta) = N_d(\mu + w\beta\Delta, w\Delta) \underset{w}{\wedge} GIG(\lambda, \delta^2, \kappa^2). \quad (4)$$

In (4) the domain of variation of the parameters is given by $\lambda \in \mathbf{R}$, $\delta \geq 0$, $\alpha \geq 0$, $\mu \in \mathbf{R}^d$, $\beta \in \mathbf{R}^d$, Δ is a positive definite $d \times d$ matrix with determinant $|\Delta| = 1$ and $\kappa^2 = \alpha^2 - \beta\Delta\beta^*$. Furthermore, $N_d(\mu, \Sigma)$ denotes the d-dimensional normal distribution with mean μ and variance Σ and $GIG(\lambda, \chi, \psi)$ denotes the generalized inverse Gaussian distribution with probability density

$$\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} w^{\lambda-1}\exp\{-1/2(\chi w^{-1} + \psi w)\}, \qquad w > 0.$$

The probability density function of the generalized d-dimensional hyperbolic distribution $H_d(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ is given by the following rather complicated expression:

$$\frac{(\kappa/\delta)^\lambda}{(2\pi)^{d/2}K_\lambda(\delta\kappa)} \cdot \frac{K_{\lambda-d/2}(\alpha\sqrt{\delta^2 + (x-\mu)\Delta^{-1}(x-\mu)^*})}{(\sqrt{\delta^2 + (x-\mu)\Delta^{-1}(x-\mu)^*}/\alpha)^{d/2-\lambda}} \exp\{\beta\cdot(x-\mu)\}, \quad x \in \mathbf{R}^d. \quad (5)$$

When evaluating the fit of a d-dimensional distribution to a particular set of data, one often makes a number of plots which compare the observed one-dimensional marginal distributions and some of the observed one-dimensional conditional distributions with the similar marginal and conditional distributions calculated from the fitted d-dimensional distribution. As shown in BLÆSILD (1981), the class of generalized hyperbolic distributions has the convenient property that it is closed under margining, conditioning and, furthermore, under regular affine transformations.

Setting $\lambda = (d + 1)/2$ in (5) one obtains the class of d-dimensional hyperbolic distributions. Using the identity $K_{1/2}(x) = (\pi/(2x))^{1/2}e^{-x}$ formula (5) turns into

$$\frac{(\kappa/\delta)^{(d+1)/2}}{(2\pi)^{(d-1)/2}2\alpha K_{(d+1)/2}(\delta\kappa)} \cdot \exp\{-\alpha\sqrt{\delta^2 + (x-\mu)\Delta^{-1}(x-\mu)^*} + \beta\cdot(x-\mu)\} \quad (6)$$

Formula (1) is obtained from (6) by setting $d = 1$, $\alpha = (\phi + \gamma)/2$, $\beta = (\phi - \gamma)/2$ and $\Delta = 1$.

The two-dimensional hyperbolic distribution, which we use for describing the joint distribution of wave height and wave period, is obtained from (6) by setting $d = 2$. Using the formula $K_{3/2}(x) = (1 + 1/x)(\pi/(2x))^{1/2}e^{-x}$ the probability density function of the two-dimensional hyperbolic distribution becomes

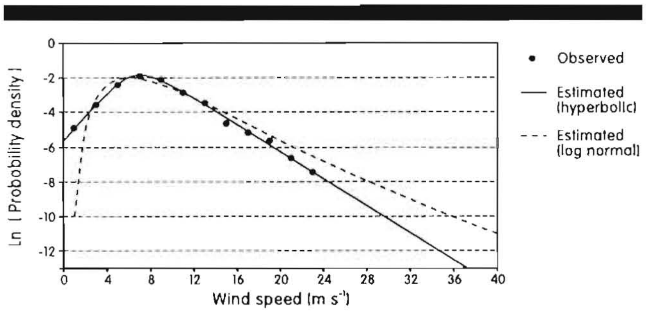

Figure 4. Hyperbolic and log-normal distributions fitted to observed wind data. Data measured at Fornæs Lighthouse at 3-hour intervals during 1990–1991.

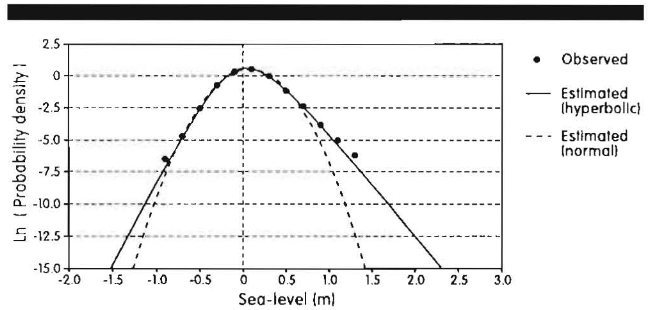

Figure 5. Hyperbolic and Gaussian distributions fitted to observed sea-level data measured at Aarhus harbour at ¼-hours intervals during 1990.

$$\frac{\kappa^3}{2\pi\alpha(1+\kappa\delta)}\exp\{-\alpha\sqrt{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)^*} + \beta\cdot(x-\mu)\}, \qquad x \in \mathbf{R}^2. \tag{7}$$

If (X_1, X_2) is distributed according to (7) it follows from BLÆSILD (1981) that the marginal distribution of X_1 and the conditional distribution of X_2 given $X_1 = x_1$ are, respectively,

$$X_1 \sim H_1(\lambda^{\#}, \alpha^{\#}, \beta^{\#}, \delta^{\#}, \mu^{\#}, \Delta^{\#}),$$

where

$$\begin{aligned}
\lambda^{\#} &= 3/2\\
\beta^{\#} &= \beta_1 + \beta_2\Delta_{21}\Delta_{11}^{-1}\\
\mu^{\#} &= \mu_1\\
\alpha^{\#} &= \Delta_{11}^{-1/2}\sqrt{\alpha^2 - \beta_2(\Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12})\beta_2}\\
\delta^{\#} &= \Delta_{11}^{1/2}\delta\\
\Delta^{\#} &= 1
\end{aligned} \tag{8}$$

and

$$X_2 | X_1 = x_1 \sim H_1(\lambda_{2.1}, \alpha_{2.1}, \beta_{2.1}, \delta_{2.1}, \mu_{2.1}, \Delta_{2.1}),$$

where

$$\begin{aligned}
\lambda_{2.1} &= 1\\
\beta_{2.1} &= \beta_2\\
\mu_{2.1} &= \mu_2 + (x_1 - \mu_1)\Delta_{11}^{-1}\Delta_{12}\\
\alpha_{2.1} &= \alpha\Delta_{11}^{1/2}\\
\delta_{2.1} &= \Delta_{11}^{-1/2}\sqrt{\delta^2 + (x_1 - \mu_1)\Delta_{11}^{-1}(x_1 - \mu_1)}\\
\Delta_{2.1} &= \Delta_{11}(\Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12}).
\end{aligned}$$

Note that these results imply, that the conditional distributions of the two-dimensional hyperbolic distribution are one-dimensional hyperbolic distributions ($d = 1$ and $\lambda = 1$) in contrast to the marginal distributions which are generalized hyperbolic distributions ($d = 1$ and $\lambda = 3/2$).

METHODS

Estimation of the parameters of the hyperbolic distribution was based on the likelihood method. Different programs were used according to whether the data was grouped or ungrouped. Recall, that a set of data is ungrouped if all the single observations are recorded. In contrast, a grouped set of data consists of the number of single observations in the different groups (intervals) into which the sample space is divided. For grouped data the estimates of the parameters were obtained by the use of the SAHARA program (CHRISTIANSEN and HARTMANN, 1988). For ungrouped data and for fitting the two-dimensional hyperbolic distribution the HYP program (BLÆSILD and SØRENSEN, 1992) was used. Copies of these programs are available from the present authors. Least-square estimation of the parameters of the one-dimensional hyperbolic distribution is discussed in McARTHUR (1987).

RESULTS AND DISCUSSION

One Dimensional Analysis

We have used the hyperbolic distribution on three types of data from the coastal environment: (1) 2 years of observations of wind velocity (recorded by the Danish Meteorological Institute at Fornæs Lighthouse as 10 min's average of measurements every 3 hours); (2) 1 year observation of sea-level recorded every quarter of an hour at Aarhus Harbour; and (3) 2 years of observations from the Great Belt of both significant wave height and wave period (T_Z, zero down crossing) (recorded 20 min every 3 hours with a logging frequency of 2.56 Hz).

It can be seen from Figures 4–7 that for all types of data a good agreement exists between observations and the fitted hyperbolic distributions. This is specially the case when one takes into consideration that the y-axis is logarithmic. Therefore, even small deviations between data and the fitted distributions will look as if the agreement is not too good. As an example, the apparent deviation in Figure 4 between the observed frequency of wind speed 18 m/sec (0.72%) and the calculated frequency (0.52%) is only small with a difference of 0.2%.

The good agreement between observations and fitted hyperbolic distribution in Figure 4 provides for good possibilities for predictions/hindcasts. According to the distribution, a wind speed of 36 m/sec has a return period of 100 years (ln probability density = −12.5). This is a slightly higher wind speed than the maximal recorded wind speed of 33 m/sec at

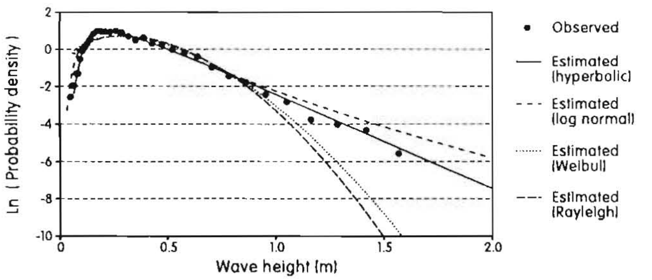

Figure 6. Distributional fits to wave heights measured in the Great Belt at 3-hour intervals during 1989–1991.

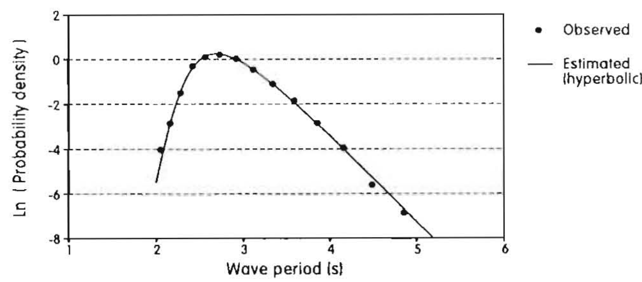

Figure 7. Hyperbolic distribution fitted to wave periods measured in the Great Belt at 3-hour intervals during 1989–1991.

the lighthouse during the last 100 years. One possible reason for a higher hindcast is that the present wind climate (1960–1990) generally is more windy compared to the 1930–1960 wind climate (KRISTENSEN and FRYDENDAHL, 1991).

Figure 5 shows the fit to sea-level data using both the Gaussian and the hyperbolic distribution. The observations are nearly symmetrical around a sea-level of 0.04 m and have negative values. Therefore the log-normal distribution would not give an adequate fit. The Gaussian distribution seems to underestimate the frequency of the extremes. The highest sea-level measured at Aarhus Harbour (Figure 1) during the last 100 years is 1.9 m o. DNN (Danish Ordnance Datum). According to the Gaussian distribution, a sea-level of 1.4 m has a return period of 100 years; whereas, the hyperbolic distribution would give a sea-level of 2.3 m with a return period of 100 years. The hyperbolic distribution thus seems to give too high sea-levels for such long return periods. However, this is somewhat in accordance with the present sea-level trend. Yearly mean sea-level is rising with a rate of 0.6 mm/yr and yearly maximum sea-level is rising with a rate of 10 mm/yr (CHRISTIANSEN *et al.,* 1992). This possibly explains that the predicted maximum sea-level for a return period of 100 years, based on the hyperbolic distribution, tends to be overestimated, in that the present prediction is based on data from the very end of the 100 year period.

Figure 6 shows that for our observations the Rayleigh as well as the Weibull distributions have difficulties in fitting frequencies of wave heights. The log-normal distribution has a good fit to small and average wave heights, whereas this distribution seems to overestimate high wave heights. The hyperbolic distribution gives a good fit in the central part of the observations as well as in the extremes. The good fit in the extremes is essential for the use of the distribution as a prediction tool. Also wave periods (Figure 7) can be fitted with high precision using the hyperbolic distribution.

Two Dimensional Analysis

Several authors, *e.g.,* BRETSCHNEIDER (1959) and BATTJES (1971) have discussed the joint distribution of individual wave heights and periods. They only considered the possibility of a linear correlation between height and period squared. HARRIS (1972) argued from theories of wave generation and wave mechanics that the highest waves correspond to intermediate periods. The wave height increases with an increase in the wave period up to the maximum height and then decreases as wave period increases farther. HARRIS (1972), therefore, found that a higher correlation generally was obtained when a parabolic function is assumed to relate wave height with the wave period. In a physical sense, this also appears more logical as for short periods, waves of large height break and do not occur; and for very long periods, there is not enough fetch or duration to develop the very high waves. It appears even more logical from a physical point of view that the above two tendencies should be free to differ from each other. This is allowed when the hyperbolic function is used to relate wave height and period.

Only one set of long-term observations of wave heights and periods are available to us. These observations come from the Great Belt, the narrow sound connecting the Kattegat and the Baltic (Figure 1). It can be seen from Figure 8 that a bivariate plot of height and period has a boomerang-like shape; and for small wave heights of approximately 0.25 m, the period can vary between 2 and 7 sec. The reason for this shape is that the observed waves partly consist of locally generated wind waves and partly of swells coming in from the Kattegat. Some of the observations might also be ship generated waves as the measuring position is very close to the most heavily trafficked ferry connections in Denmark.

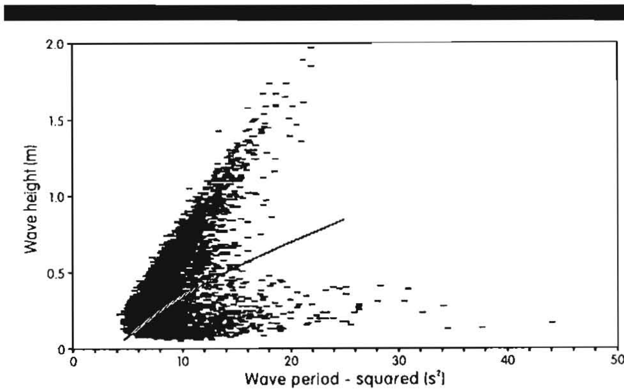

Figure 8. Bivariate plot of wave heights and wave periods squared (Great Belt, 1989–1991). The solid line represents the wave height-period combinations for the longest local fetch.

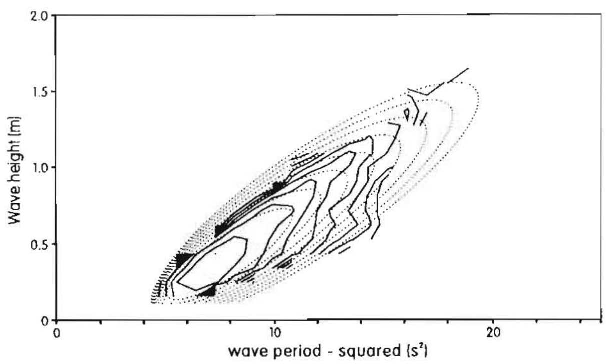

Figure 9. Equidistant contours of the logarithm of the estimated probability density function, for the joint distribution of wave heights and periods (dotted), and the corresponding contours of the observed distribution (unbroken).

We have tried to filter some of this "noise" away. Using the wave prediction formula in CERC (1975), the solid line in Figure 8 shows the possible wave height-period combinations for the longest fetch. Observations above this line are then considered to be locally generated wind waves.

For these waves, we have considered the two-dimensional hyperbolic distribution as a model for the joint distribution of wave height and period squared. The parameters of the two-dimensional hyperbolic distribution have been estimated using the HYP program (BLÆSILD and SØRENSEN, 1992). This program also produces the plots in Figure 9 and Figure 10 from which the agreement between the observed and the estimated distribution may be evaluated. The dotted curves in Figure 9 are equidistant contours of the logarithm of the probability density function of the estimated distribution, and the unbroken curves are the corresponding contours of the observed distribution. The estimated marginal distributions of wave height and period squared, respectively, are obtained from the estimated two-dimensional distribution using (8). Figure 10 shows the logarithmic probability density functions of the estimated marginal distributions together with the observed marginal distributions. Figures 9 and 10 indicate a reasonable agreement between the observed and the estimated joint distribution of wave height and period squared.

As mentioned after Formula (8), a marginal distribution of a two-dimensional hyperbolic distribution is a generalized hyperbolic distribution ($d = 1$ and $\lambda = 3/2$) and not a hyperbolic distribution ($d = 1$ and $\lambda = 1$). The fact that the distribution of, for instance, wave height earlier in the paper has been described by a one-dimensional hyperbolic distribution may therefore seem contradictory to the fact that the marginal distribution of wave height in the two-dimensional model is not hyperbolic. However, for suitable choices of parameters the probability density functions of the generalized hyperbolic distribution with $d = 1$ and $\lambda = 3/2$ and of the hyperbolic distribution ($d = 1$ and $\lambda = 1$) are nearly identical.

The two-dimensional plot in Figure 9 shows the joint distribution of wave height and period squared. However, as the

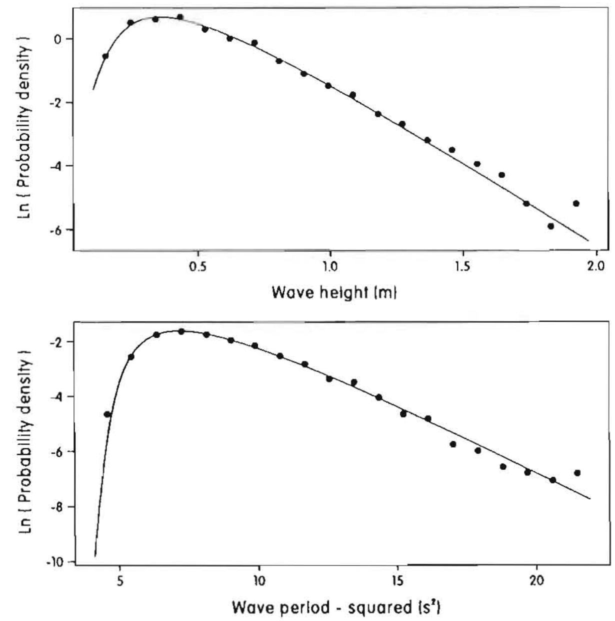

Figure 10. The logarithmic probability density function of the estimated marginal distributions of wave heights and wave periods, respectively (lines) and the observed marginal distributions (markers).

depth at the measuring station is 34 m and the observed periods are in the range of 2–5 sec, this means that the observed waves all are short, deep water waves. For such waves $L \approx 1.56\ T^2$ (in the MKS-system). Therefore, the horizontal axis in Figure 9 corresponds to L/1.56 which means that we also are given a joint distribution of wave-height and -length. Wave steepness, defined as $\delta = H/L$, is therefore also expressed in this distribution, and we may express the one-dimensional hyperbolic distribution of the wave steepness for a given height, length or period. In relation to the design of off-shore constructions, the wave steepness is an important parameter.

CONCLUSIONS

We find it important to note that many of the distributions traditionally used to evaluate wave and sea-level data are limits of the hyperbolic distribution. Our findings suggest that the hyperbolic distribution, because of its higher flexibility (being a four parameter distribution), provides for better description of the long-term probability density distribution of the one-dimensional observations.

Given the nature of our joint observations of wave height and period, we consider our findings on the two-dimensional distribution encouraging. We hope that more progress on this matter will become available as data from other locales are treated with the same approach. This would provide a better possibility for finding the critical wave period values associated with a given design wave height.

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