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Time Net for Ground-Water Flow in Idealized Coastal Wedge

H.L. Vacher, T.A. Farkas, and J.L. Robinson

Department of Geology
University of South Florida
Tampa, FL 33620, USA

ABSTRACT

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The classic Glover model for head and stream function in an indealized coastal wedge of fresh ground water in a homogeneous and isotropic aquifer is extended to estimate the length of time for fresh ground water to flow to the outflow face ("exit time"). An analytical solution is given for the exit time (τ) for water that travels the streamline that lies at sea level, and τ's for water traveling deeper streamlines are found numerically by summing incremental travel times as water is tracked incrementally along a streamline. Results are given by dimensionless "time nets," which show streamlines and contours of dimensionless exit time, and thereby allow estimation of exit time for many combinations of hydraulic conductivity, porosity and fresh-water discharge. The model ignores the presence of an underlying transition zone and, therefore, applies only to the part of the wedge that lies shoreward of the effects of the lower boundary on the configuration of the interface and other isochlors.

ADDITIONAL INDEX WORDS: *Coastal hydrology, coastal ground water, ground-water models, ground-water flow nets.*

INTRODUCTION

Fresh ground water near the coast typically occurs in a wedge-shaped body in which the land-derived fresh ground water flows seaward to a discharge face along the shoreline. The fresh-water wedge overlies saline water of marine derivation. How long does it take for a parcel of fresh water to flow from a given location in the fresh-water wedge to the point of shoreline discharge? We will call this the "exit time" for that location.

Intuitively, it can be expected that this exit time will depend on such factors as the location of the point in question, the quantity of fresh water flowing through the wedge, and the hydraulic characteristics of the aquifer (hydraulic conductivity, porosity). It can also be anticipated that real-world cases could be quite complex because of geologic heterogeneities affecting the spatial variation of porosity, hydraulic conductivity and anisotropy. If the location of the fresh-water/salt-water interface is known *a priori*, and the geology and its variation are also known, then the distribution of head and various travel times can be calculated using standard numerical flow models (*e.g.*, MERCER and FAUST, 1981; BEAR and VERRUIJT, 1987), by assuming that the interface is the lower flow boundary. If the location of the interface must be found as a part of the solution, then a Dupuit-Ghyben-Herzberg-type numerical model may be appropriate (*e.g.*, VOSS, 1984a). If allowance is made for the fact that the "interface" is probably a transition, or mixing, zone, then the problem requires a model coupling density-dependent flow and solute transport (*e.g.*, VOSS, 1984b; HUYAKORN *et al.*, 1987), and other aquifer properties, such as dispersivity, must also be known; this has been done at a few places (*e.g.*, ANDERSEN *et al.*, 1988). But, for many investigations, such numerical modeling is "overkill" (MERCER and FAUST, 1981, p. 6) or premature, because "the objectives may require only a very simple model . . . (or) lack of data may not justify a sophisticated model" (MERCER and FAUST, 1981, p. 4). For many purposes, " . . . a one-fluid model . . . is often the type which should be used as it is based on the simplest set of assumptions

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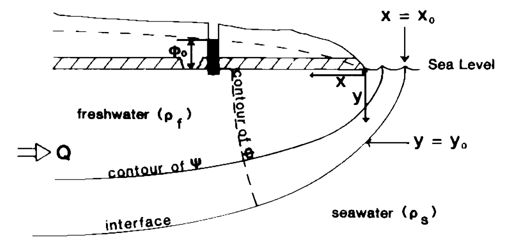

Figure 1. Definition of the coastal flow system of Glover (1959, 1964).

and does not tend to over-model or over-represent the hydrologic system" (VOSS, 1984a, p. 6).

The simplest models treat idealized cases and are given in the form of analytical solutions to underlying flow equations. These models have the distinct advantage that unknowns can be calculated from equations or read from charts or graphs. Standard textbooks include many examples (FREEZE and CHERRY, 1979; TODD, 1980; WALTON, 1985; FETTER, 1988). One of the classic analytical models is that by GLOVER (1959, 1964) for the coastal wedge. Equations are given by GLOVER (1959, 1964) for the head and stream function for the idealized case of steady-state fresh-water flow in a homogeneous, isotropic aquifer with a sharp interface between the fresh water and underlying static seawater. Contours of these equations give the familiar flow net for the coastal wedges (equipotentials and streamlines, respectively) that is reproduced in several textbooks. In the following, we extend the Glover model to include exit time. The resultant time net (contours of exit time and stream function) can be used to assess travel times in idealized coastal wedges for the same general purposes and subject to the same limitations of underlying assumptions that the flow nets of the original Glover model can be used to assess the distribution of head.

GLOVER'S SOLUTION FOR FRESH-WATER WEDGE

Basic Equations

The flow system and nomenclature are shown in Figure 1. The densities of the fresh ground water and seawater are ρ_f and ρ_s, respectively. The wedge is confined by an aquiclude at sea level, so fresh-water discharge (Q in m^3/day/m parallel to shore) is uniform across the section. Hydraulic conductivity (K, in m/day) of the aquifer is homogeneous and isotropic. The origin of the x,y coordinates is at the shoreline, with x increasing landward, and y downward. There is a nonzero outflow face, with its outer edge at $y = 0$ and $x = x_0$. Accordingly, there is a nonzero fresh-water thickness at the shoreline (y_0).

Glover's solutions to the underlying Laplace equations are

$$\phi^2 = (Q/K\alpha)[x + (x^2+y^2)^{1/2}] \qquad (1)$$

$$\psi^2 = (Q/K\alpha)[-x + (x^2+y^2)^{1/2}] \qquad (2)$$

where ϕ is the head, ψ is the stream function, and α is the density-difference ratio, $\rho_f/(\rho_s - \rho_f)$. The head at the base of the confining bed (ϕ_0), from equation 1 and $y = 0$ is

$$\phi_0{}^2 = (2Q/K\alpha)x \quad . \qquad (3)$$

Within 2 percent, ϕ_0 gives the position of the water table that would be present if the confining bed were absent (CHARMONMAN, 1965).

Because the constant value of $K\psi$ along a particular streamline is numerically equal to the ground-water discharge above that streamline,

$$mQ = K\psi \quad , \qquad (4)$$

where m is the proportion of the total ground-water discharge that passes above the streamline. Then, from equations 2 and 4, the equation for the streamline underlying mQ is

$$y_m = (2m^2Q\alpha/K)\,x + (m^2Q\alpha/K)^2 \quad . \qquad (5)$$

From equation 5, and because $m = 1$ along the interface, the position of the interface is given by

$$y_1{}^2 = (2Q\alpha/K)\,x + (Q\alpha/K)^2 \quad . \qquad (6)$$

From equation 6 and $y = 0$, the position of the outer edge of the outflow face is

$$x_0 = -Q\alpha/2K \quad . \qquad (7)$$

Finally, from equation 6 and $x = 0$, the thickness of the wedge at the shoreline is

$$y_0 = Q\alpha/K \quad . \qquad (8)$$

Example wedges drawn from Glover's equations are shown in Figure 2. It is clear in these

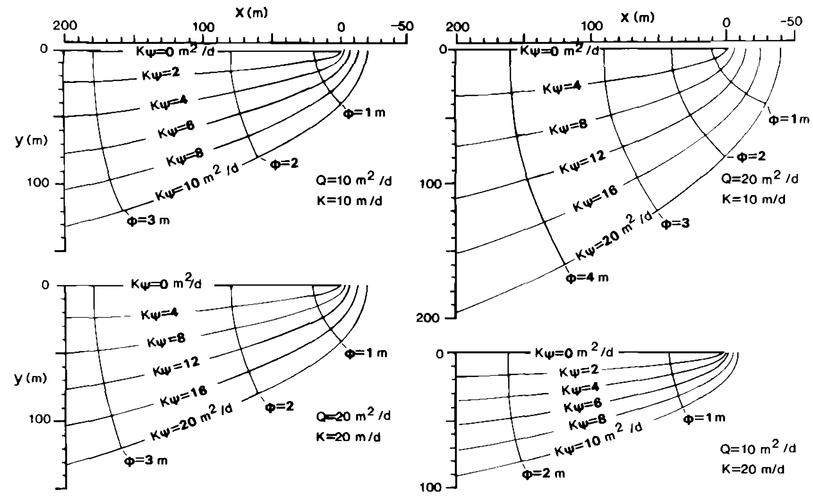

Figure 2. Flow nets for four coastal wedges. The two on the left have the same Q/K and, therefore, the same geometry. The wedges on the right have a different geometry in accordance with the different Q/K.

examples that Q/K controls the shape of the wedge. Wedges with large discharge or low hydraulic conductivity are relatively inflated. Such wedges also have relatively large outflow faces.

Dimensionless Solution

The examples of Figure 2 are particular cases of the general solution shown by the dimensionless flow net of Figure 3. Glover's analytical model can be transformed to dimensionless form by

$$x^* = x/y_0 = Kx/Q\alpha \tag{9}$$

$$y^* = y/y_0 = Ky/Q\alpha \tag{10}$$

$$\phi^* = K\phi/Q = \phi\alpha/y_0 \tag{11}$$

$$\psi^* = K\psi/Q = \psi\alpha/y_0 \quad , \tag{12}$$

where x^* and y^* are the dimensionless coordinates, ϕ^* is dimensionless head, ψ^* is the dimensionless stream function, and the second equality in equations 9–12 is from equation 8. Then, equations 1 and 2 become

$$(\phi^*)^2 = x^* + [(x^*)^2 + (y^*)^2]^{1/2} \tag{13}$$

$$(\psi^*)^2 = -x^* + [(x^*)^2 + (y^*)^2]^{1/2} \quad , \tag{14}$$

respectively.

Equations 13 and 14 indicate that the flow net and the configuration of the wedge shown in Figure 3 are independent of Q and K. For example, the edge of the outflow face occurs at $x^* = -0.5$. The location of the interface at the shoreline is at $x^* = 0$ and $y_0^* = 1$. Streamlines are given by

$$(y_m^*)^2 = 2m^2 x^* + m^4 \quad , \tag{15}$$

where m is ψ^*, and the interface is at

$$(y_1^*)^2 = 2x^* + 1 \quad . \tag{16}$$

The head at the base of the confining bed is

$$\phi_0^* = (2x^*)^{1/2} \quad . \tag{17}$$

TRAVEL TIMES AND EXIT TIMES

Consider flow along a streamline from (x_1^*, y_1^*) to (x_2^*, y_2^*). Let t_1^* and t_2^* represent the dimensionless age of the water at the respective points, where dimensionless ages (t^*) are related to ages (t) by

$$t^* = Kt/ny_0\alpha \tag{18}$$

and n is porosity. Then, the dimensionless travel time between the two points is

$$\Delta t^* = t_2^* - t_1^* \tag{19}$$

If (x_2^*, y_2^*) is located on the outflow face, Δt^* is

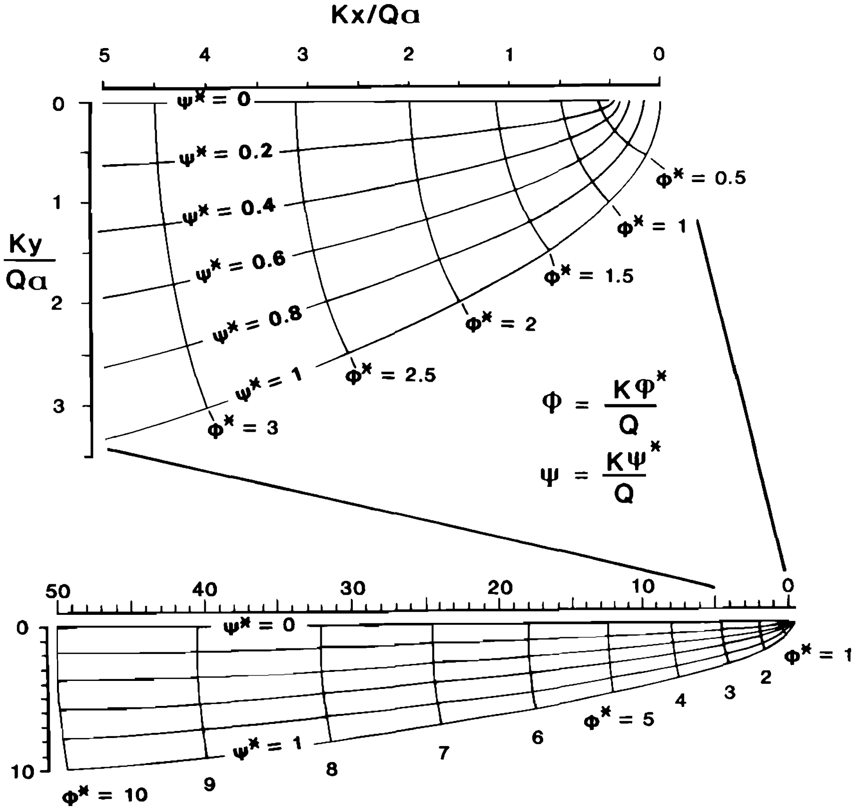

Figure 3. Dimensionless flow net for the coastal wedge with homogeneous and isotropic K. The upper net is like that of Glover (1959, 1964) and covers the same dimensional area as the wedge in Figure 2 where Q/K = 1. The lower net expands the coverage to a larger area, which would apply for small Q/K and a large depth to the underlying boundary.

the dimensionless exit time, τ^*, that is, the time for the water to flow from (x_1^*, y_1^*) to the point of discharge from the wedge.

Along $y^* = 0$

From equations 9 and 18, the dimensionless interstitial velocity (v^*) along $y^* = 0$ is

$$v_x^* = dx^*/dt^* = (n\alpha/K)\, v_x \qquad (20)$$

where the component v_x^* is the total velocity because flow is parallel to x^*. From Darcy's Law, which is

$$v_x = -(K/n)\partial\phi/\partial x \quad , \qquad (21)$$

and equations 9, 11, and 20, it follows that

$$v_x^* = -\partial\phi^*/\partial x^* \qquad (22)$$

Then, from

$$\Delta t^* = \int_{t_1^*}^{t_2^*} dt^* = \int_{x_1^*}^{x_2^*} (1/v_x^*)\, dx^* \qquad (23)$$

and equations 17 and 22, the dimensionless travel time between (x_1^*, y_1^*) and (x_2^*, y_2^*) is

$$\Delta t^* = \frac{2(2)^{1/2}}{3}\left[(x_1^*)^{1/2} - (x_2^*)^{1/2}\right] \quad . \qquad (24)$$

From equation 24 with $x_2^* = 0$ and $x^* = x_1$, the dimensionless exit time is

$$t^* = (2/3)\,(2)^{1/2}\,(x^*)^{3/2} \quad . \qquad (25)$$

As seen in equations 24 and 25, dimensionless travel times and exit times depend only on location and not on Q, K, or n. These latter parameters affect the dimensional travel times and exit times:

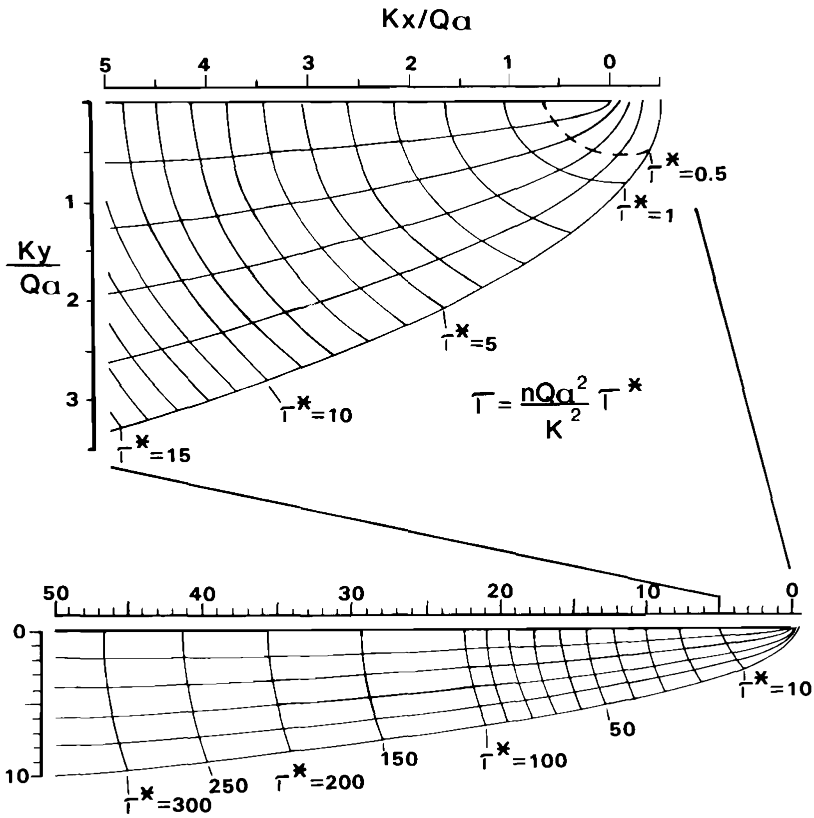

Figure 4. Dimensionless time net, showing streamlines and contours of exit time.

$$\Delta t = (2n/3)\,(2\alpha/KQ)^{1/2}\,(x_1^{3/2} - x_2^{3/2}) \quad (26)$$

and

$$\tau = (2n/3)\,(2\alpha/KQ)^{1/2} x^{3/2} \quad (27)$$

which follow from equations 8, 9, 18, 24 and 25.

Below y = 0

Exit times and travel times within the wedge can be found numerically by summing travel-time increments along a streamline. The computational scheme used here is similar in principle to that of JAVANDEL *et al.* (1984) and BEAR and VERRUIJT (1987). In terms of dimensionless variables and the coastal-wedge problem, the streamline is identified by its stream function which is calculated from equation 14. This gives m, which is used with equation 15 and $x^* + \Delta x^*$ to find $y^* + \Delta y^*$ for the upstream point of each increment. The travel distance, Δs^*, along the increment is found from the Pythagorean relation and (x^*, y^*) and $(x^* + \Delta x^*, y^* + \Delta y^*)$. This distance, combined with ϕ^* at (x^*, y^*) and $(x^* + \Delta x^*, y^* + \Delta y^*)$, gives the average velocity, v^*_{ave}, across the increment by Darcy's Law. The time increment, then, is $\Delta s^*/v^*_{ave}$. The sum of the time increments from the outflow face to a given point inland gives the exit time for that point.

Results

The results of the calculations are shown in Figure 4 as a dimensionless time net showing contours of exit time (τ^*) and stream function. The Δx^* increment for the calculations was 0.1. With this increment, the numerically calculated τ^* along $y = 0$ agrees with the value calculated from the analytical expression (equa-

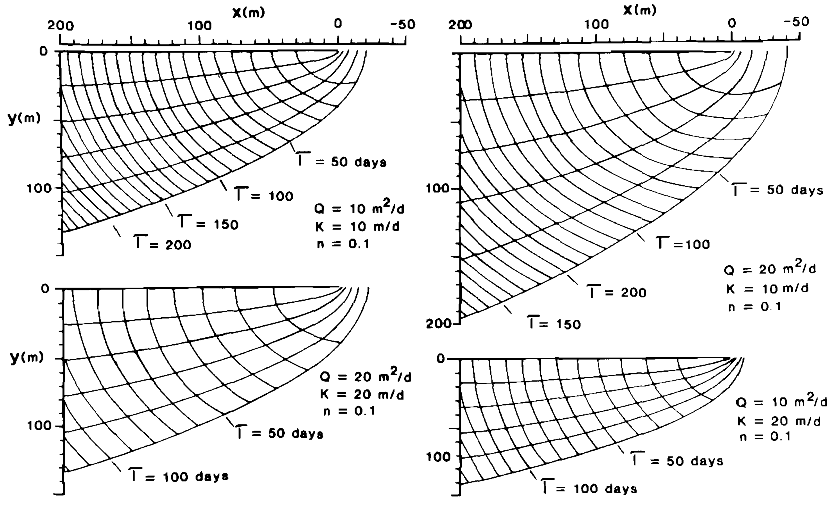

Figure 5. Time nets for the four coastal wedges of Figure 2.

tion 25) to within 1 percent for $x^* > 1$ and to within about 0.1 percent for $x^* > 5$.

Dimensional time nets for the examples of Figure 2 are shown in Figure 5 and illustrate the effect of the controlling variables, Q and K. The effect of porosity, n, is linear; that is, if n were 0.2 instead of 0.1, all of the contours would be in the same place in the figures, and the times would all be doubled.

Travel times can be read directly from the time nets as the difference in exit time between two points on a streamline.

The spacing of exit-time contours illustrates the increase of ground-water velocity towards the shoreline. This is also shown in the spacing of equipotentials and streamlines in Figure 3. The shoreward increase in velocity is illustrated explicitly in Figure 6, for which contours of v^* were calculated from

$$v^* = -\nabla\phi^* \quad . \tag{28}$$

As shown in the figure, the velocity at any x^* is largest at $y = 0$ and decreases downward. Although less obvious, such differences in velocity can be seen also in the spacing of contours in Figures 3 and 4.

Discussion

The ease with which exit times can be calculated can be illustrated with an example. Let the distance from the shoreline (x) be 200 m, and the depth (y) into the wedge (below sea level) be 20 m. Assume that the conditions of Glover's derivation (*i.e.*, homogeneity, isotropy, and an interface) apply to an aquifer with $K = 100$ m/day and $n = 0.2$; finally, suppose that $Q = 20\ m^2$/day (approximately equivalent to a recharge of 0.4 m/yr between the point at x and a divide some 18 km further upstream). Then, the dimensionless horizontal distance (x^*) and depth (y^*) are 25, and 2.5 respectively, from equations 9 and 10, respectively. From Figure 4, the dimensionless exit time (τ^*) is read off as 120; therefore, the exit time is 77 days, from equation 18. At the same time, the depth to the interface at x is 57 m (from equation 6, or equations 16 and 10), and 35 percent of the flow is above y (equation 15).

CONCLUDING REMARKS

The Glover model assumes that the freshwater wedge rests on static seawater. The model, therefore, does not take account of tidal fluctuations and the presence of an underlying transition zone. Although numerical models have been developed that can treat coastal wedges with dispersion about the interface, we know of no study that explicitly determines exit time in such systems. It is possible, however, to make some generalized comments on differences that can be expected regarding exit time.

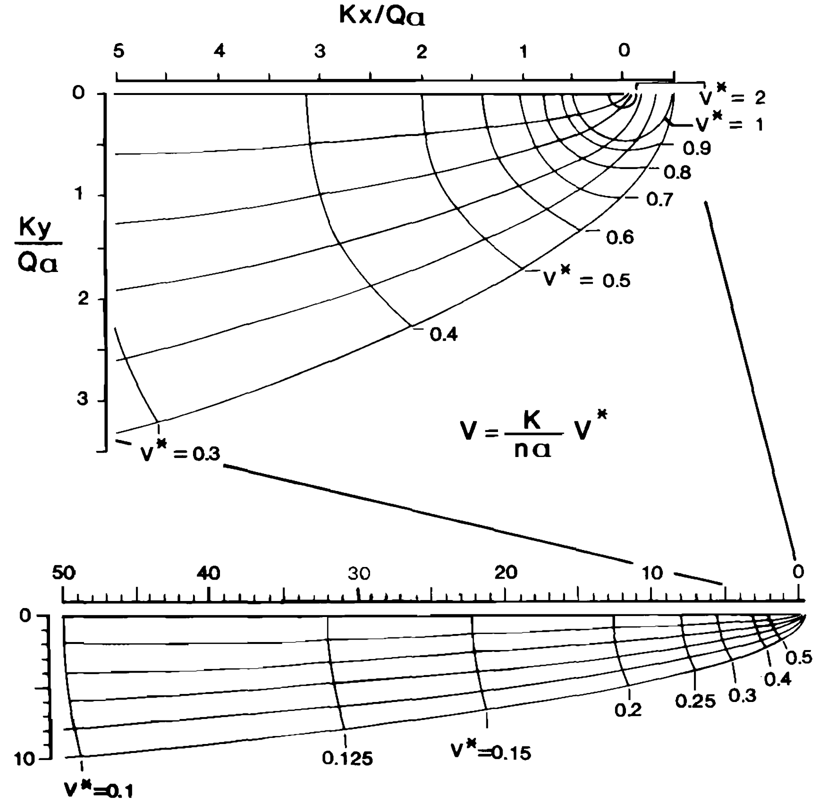

Figure 6. Dimensionless velocity cross section.

One effect of the brackish discharge from the transition zone is the generation of a cyclic circulation of seawater (COOPER, 1959, 1964). This means that the salt-water head must be less than zero, and so, from the Hubbert equation (HUBBERT, 1940), the interface and other streamlines would be deeper than calculated from interface models (VACHER, 1988). Exit times, therefore, would be somewhat larger.

A second feature of the transition zone is that isochlors within the transition zone must be perpendicular to the underlying, low-permeability boundary of the aquifer. As a result, the concave-upward interface that is portrayed by the Glover model is succeeded by a more landward sector in which isochlors are convex upward as they curve downward to meet the boundary condition at the base of the aquifer (KOHOUT, 1960, 1964; BEAR, 1979). The Glover model would not be appropriate in this landward sector.

In view of these two features of the transition zone, it follows that the exit times given here for steady-state, homogeneous- and isotropic-K fresh-water wedges are best considered as lower limits of exit times in the region for x* less than that where the geometry of the wedge is affected by the lower boundary of the aquifer.

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□ RÉSUMÉ □

On a étendu le modèle classique de Glover à l'étude en mileu côtier d'un coin d'eau douce souterraine dans une nappe homogène et isotropique, afin d'estimer la longueur de temps nécessaire à l'eau douce souterraine pour s'écouler vers la face externe (temps d'évacuation). Une solution analytique est donnée pour ce temps d'évacuation ("exit time") désignant l'eau qui parcourt le fil du courant au niveau de la mer et (π) l'eau parcourant le courant à des niveaux plus profonds. On peut les trouver numériquement en faisant la somme des accroissements de temps de parcours alors que l'eau est suivie en accroissement dans le sens du courant. Les résultats sont donnés en "réseaux de temps" adimensionnels qui montrent les orientations du courant et les contours adimentionnels du temps d'évacuation, et permettent donc l'estimation du temps de sortie pour de nombreuses combinaisons de conductivité hydraulique, porosité et débit d'eau douce. Le modèle ignore la présence d'une zone de transition sous-jacente et ne s'applique donc qu'à la partie du coin située côté mer des effets de l'environnement sous-jacent, sur la configuration de l'interface et autres isochlores.—*Catherine Bressolier-Bousquet, Géomorphologie EPHE, Montrouge, France.*