The object of this column is to enhance our readers' collections of interesting and novel problems in chemical engineering. We request problems that can be used to motivate student learning by presenting a particular principle in a new light, can be assigned as novel home problems, are suited for a collaborative learning environment, or demonstrate a cutting-edge application or principle. Manuscripts should not exceed 14 double-spaced pages and should be accompanied by the originals of any figures or photographs. Please submit them to Dr. Daina Briedis (e-mail: briedis@egr.msu.edu), Department of Chemical Engineering and Materials Science, Michigan State University, East Lansing, MI 48824-1226.

NATURAL CONVECTION IN ENCLOSED POROUS OR FLUID MEDIA

Esteban Saatdjian, François Lesage, and José Paulo B. Mota[1]
ENSIC - Ecole Nationale Supérieure des Industries Chimiques • 54001 Nancy Cédex, France
[1] Requimte/CQFB, Universidade Nova de Lisboa • 2829-516 Caparica, Portugal

Computational Fluid Dynamics (CFD) is currently an elective, senior/graduate-level course in chemical and mechanical engineering departments. Our teaching experience in this course has shown us that students learn more (and appreciate the course better) when the grading is based on the completion of a computer project (which is time consuming for both the student and the instructor) instead of a formal exam. For a first course, however, the CFD instructor must take great care in choosing a project to be completed by students during one semester. Most of the students enrolled in an introductory CFD course have little experience whatsoever in the subject and can only start the project after learning the basic algorithms used in CFD. This implies that complex projects involving turbulence, two-phase flow in 3-D, or complex geometries are not appropriate at this level. We have found that a transport phenomena project, involving the numerical solution of coupled transport equations, without turbulence, in 2-D is a good choice for such a project. The students thus learn many of the important aspects of CFD such as the discretization of parabolic and elliptic equations, solution of a tri-diagonal linear system of equations using either direct or implicit methods, and the treatment of boundary conditions.

One such project[1] is the study of natural convection in a porous media bounded by two concentric, horizontal cylinders. This corresponds to the thermal insulation of a horizontal, cylindrical tube with glass-wool. In a previous paper,[1] the numerical solution showed that when the Rayleigh number is high enough, a secondary cell appears in the top part of the porous layer. In that region the temperature gradient and the gravity vector are in opposite directions, leading to instability there. On the other hand, in the bottom section of the layer, the gravity vector and the temperature gradient are in the same direction. The bottom region is thus stable and no secondary cells form there. The additional recirculation cell formed leads to an increase in the average Nusselt number and thus to an increase in the rate of energy loss to the surroundings. An industrial application of the problem considered here could

***Esteban Saatdjian** has been a professor of chemical engineering at ENSIC in Nancy, France, since 1989. He teaches transport phenomena and computational fluid dynamics (CFD) to undergraduate and graduate students. His current research interests are in mixing of viscous fluids, magnetohydrodynamic flows, and micro-fluidics. He graduated from Rensselaer Polytechnic Institute in chemical engineering in 1975 and obtained his Ph.D. at the University of Paris in 1980.*

***François Lesage** is an associate professor of chemical engineering at ENSIC. He teaches process control, CFD, and numerical methods to undergraduate students. His research interests are in the domain of process optimization and two-phase flow.*

***José Paulo Mota** has been a professor of chemical engineering at Universidade Nova de Lisboa, Lisbon, Portugal, since 1997. He teaches transport phenomena, process control, and optimization. His current research interests are in adsorption, mixing, and molecular simulations. He graduated from the University of Porto in 1991 and obtained his Ph.D. degree in Nancy, France.*

be a very hot, horizontal cylinder of large diameter or the fuselage of an airplane that is to be insulated with glass wool in order to minimize fuel-consuming heat losses. The question an engineer would ask is as follows: *Is it possible, with the same amount of insulation, to avoid the formation of the secondary cell and thus reduce the heat transfer losses to the surroundings at high Rayleigh numbers?*

The Rayleigh number is proportional to the third power of a reference length.[2] If the thickness of the porous layer in the top section of the layer is reduced, the local Rayleigh number there would also be reduced and it could be possible to avoid the formation of the secondary cell. To determine whether a change in geometry can lead to a reduction in energy loss, the governing equations must be solved for the case where the two cylinders are in an eccentric position with the minimum gap at the top of the layer.

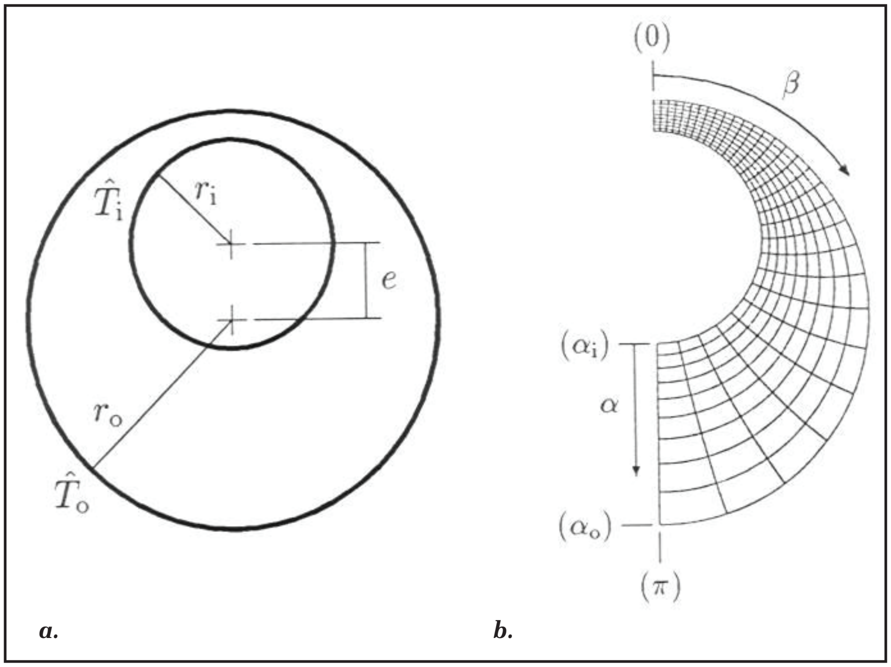

***Figure 1.** a) Geometry of the annular space between eccentric cylinders; r_i and r_o are the inner and outer radii of the two cylinders and e is the distance between the centers of the two cylinders. b) Bipolar coordinate grid.*

This new problem and other similar problems can be formulated from the study of natural convection at moderate Rayleigh numbers in enclosures. These problems concern the insulation of horizontal pipes. They are of the same difficulty as the project presented earlier.[1] Students can not only use these problems to master the basics of CFD but they may also improve their understanding of coupled heat and momentum transfer phenomena. The benefits for the instructor of the course are numerous as we shall see—one of them is being able to give a different project each semester.

The object of this paper, which is tied to our earlier paper,[1] is to first present the problem of natural convection between eccentric, porous cylinders and to determine, using CFD, whether a change in geometry can lead to a reduction in energy losses. Other tube geometries and fluid media are discussed in a later section. Throughout the years, all of these projects have been given to our senior/graduate students in this course.

NATURAL CONVECTION BETWEEN ECCENTRIC, POROUS CYLINDERS

The problem considers a porous medium bounded by two eccentric, horizontal cylinders. In order to suppress the secondary cell that forms at high Rayleigh numbers in the concentric case, the insulation thickness has to be reduced in the top part of the layer, in order to reduce the local Rayleigh number. The geometry between two eccentric cylinders (see Figure 1 a) can be described by several different coordinate systems; bipolar cylindrical coordinates are perhaps the easiest to use. If the minimum gap is at the top of the layer, the transformation equations from Cartesian coordinates (x, y) to bipolar coordinates (α, β) are:

$$x = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}, \; y = \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta} \qquad (1)$$

where a is the distance from the origin to a focal point.

In a bipolar orthogonal coordinate system, the curves of constant α and the curves of constant β are both circles that intersect at right angles. If the minimum gap is at the top of the layer then the two foci are located at coordinates (0,–a) and (0,a). Notice that $-\infty < \alpha < \infty$ and $0 < \beta < 2\pi$. Only one dimensionless parameter (the radius ratio) is necessary to define the geometry between two concentric cylinders, however two are required between eccentric cylinders. These two parameters are the radius (or clearance) ratio $\delta = r_o/r_i$, and the eccentricity ratio $\varepsilon = e/(r_o - r_i)$ where e is the distance between the centers of the two cylinders. A bipolar coordinate grid is shown in Figure 1b.

In order to write the natural convection equations in bipolar coordinates it is necessary to first review vector analysis and orthogonal curvilinear coordinates in particular.[3] If **r** is the position vector of a point P in space:

$$\mathbf{r} = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}\mathbf{i} + \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta}\mathbf{j} \qquad (2)$$

then tangent vectors to the α and β curves, $\mathbf{U}_\alpha$ and $\mathbf{U}_\beta$, at P are given by:

$$\mathbf{U}_\alpha = \frac{\partial \mathbf{r}}{\partial \alpha} = -\frac{a\sinh\alpha\sin\beta}{(\cosh\alpha-\cos\beta)^2}\mathbf{i} + \frac{a(1-\cosh\alpha\cos\beta)}{(\cosh\alpha-\cos\beta)^2}\mathbf{j} \quad (3)$$

$$\mathbf{U}_\beta = \frac{\partial \mathbf{r}}{\partial \beta} = \frac{a(\cosh\alpha\cos\beta-1)}{(\cosh\alpha-\cos\beta)^2}\mathbf{i} - \frac{a\sinh\alpha\sin\beta}{(\cosh\alpha-\cos\beta)^2}\mathbf{j}. \quad (4)$$

This coordinate system is orthogonal since $\mathbf{U}_\alpha \cdot \mathbf{U}_\beta = 0$. The scale factors h_α and h_β in the two directions are calculated by:

$$h_\alpha = \left|\frac{\partial \mathbf{r}}{\partial \alpha}\right| = \frac{a}{\cosh\alpha-\cos\beta},\ h_\beta = \left|\frac{\partial \mathbf{r}}{\partial \beta}\right| = \frac{a}{\cosh\alpha-\cos\beta}, \quad (5)$$

which are equal so for convenience from here on $h_\alpha = h_\beta = h$.

In bipolar coordinates, the 2-D conservation of energy equation for natural convection between eccentric, porous cylinders is:

$$(\rho c_p)^* \frac{\partial T}{\partial t} + (\rho c_p)_f \left[\frac{V_\alpha}{h}\frac{\partial T}{\partial \alpha} + \frac{V_\beta}{h}\frac{\partial T}{\partial \beta}\right] = \frac{\lambda^*}{h^2}\left[\frac{\partial^2 T}{\partial \alpha^2} + \frac{\partial^2 T}{\partial \beta^2}\right] \quad (6)$$

where, using the same notation as in Reference 1, $(\rho c_p)^*$ and $(\rho c_p)_f$ are the heat capacities per unit volume of the porous medium as a whole (fluid and solid) and of the fluid, respectively, and λ^* is the thermal conductivity of the porous medium.

Dimensionless quantities are noted with the superscript "+". If the velocity components and time are scaled by $\lambda^*/(\rho c_p)_f T_i$, and $(\rho c_p)^* a^2/\lambda^*$ and with $h^+ = h/a^1$, the conservation of energy equation becomes:

$$\frac{\partial \Theta^+}{\partial t^+} = \frac{1}{h^{+2}}\left(\frac{\partial^2 \Theta^+}{\partial \alpha^2} + \frac{\partial^2 \Theta^+}{\partial \beta^2}\right) - \frac{1}{h^{+2}}\frac{\partial \psi^+}{\partial \alpha}\frac{\partial \Theta^+}{\partial \beta} + \frac{1}{h^{+2}}\frac{\partial \psi^+}{\partial \beta}\frac{\partial \Theta^+}{\partial \alpha} \quad (7)$$

where $\Theta^+ = (T - T_o)/(T_i - T_o)$ is the dimensionless temperature. In the equation above, the dimensionless stream function ψ^+ was defined as:

$$\frac{a h^+ V_\alpha^+}{r_i} = -\frac{\partial \psi^+}{\partial \beta},\ \frac{a h^+ V_\beta^+}{r_i} = \frac{\partial \psi^+}{\partial \alpha} \quad (8)$$

which automatically satisfies the continuity equation.

As in Reference 1, fluid flow in the porous media is assumed to be governed by Darcy's law:

$$V = -\frac{k}{\mu}\left[\nabla P - \rho\left(1-\beta_o (T-T_o)\right)\mathbf{g}\right] \quad (9)$$

where k is the permeability of the porous medium, and β_o is the thermal expansion coefficient. The fluid density is assumed constant in all terms except when multiplied by gravity; this is the Boussinesq approximation. Taking the curl of the above equation in order to eliminate P as dependent variable, the dimensionless stream function equation is obtained:

$$\frac{1}{h^{+2}}\left[\frac{\partial^2 \psi^+}{\partial \alpha^2} + \frac{\partial^2 \psi^+}{\partial \beta^2}\right] = \frac{a}{r_i} Ra\left[\sinh\alpha\sin\beta\frac{\partial \Theta^+}{\partial \alpha} + (1-\cosh\alpha\cos\beta)\frac{\partial \Theta^+}{\partial \beta}\right] \quad (10)$$

where $Ra = (\rho c_p)_f\, k g r_i \beta_o (T_i - T_o)/\lambda^* \nu$ is the same Rayleigh number as defined in Reference 1. Eqs. (7) and (10) can be solved numerically using finite difference techniques; the developed code is very similar to the one developed in Reference 1.

As in the concentric case, the dimensionless heat flow into or out of the annular region Q can be easily calculated:

$$Q = 2\int_0^\pi \left(\frac{\partial \Theta^+}{\partial \alpha}\right) d\beta \text{ at } \alpha = \alpha_i \text{ or } \alpha = \alpha_o. \quad (11)$$

At steady state, the heat flow into or out of the layer must be the same; this criterion allows the student to ascertain whether the results are accurate enough. When the relative difference between the heat into the annulus and the heat out is above a few percent, the student can either refine the grid or set stricter convergence tests for Θ^+ and ψ^+ in the numerical program to reduce this difference.

Special care must be taken in evaluating the above integrals. In particular, second or third order (not first order) finite difference formulas should be used to reduce the truncation error. The third order difference formula for the inner cylinder temperature gradient, obtained by Taylor series expansion,[4] is:

$$\left(\frac{\partial \Theta^+}{\partial \alpha}\right)_{\alpha_{in\,j}} \doteq \frac{-11\Theta^+_{1,j} + 18\Theta^+_{2,j} - 9\Theta^+_{3,j} + 2\Theta^+_{4,j}}{6\Delta\alpha} + O(\Delta\alpha)^3. \quad (12)$$

The average Nusselt number, Nu, is usually defined as the ratio of the heat transfer by convection to the heat transfer by conduction. In bipolar coordinates the solution of the heat conduction equation $\nabla^2\Theta^+ = 0$ with the given boundary conditions leads to:

$$\Theta^+ = \frac{\alpha}{\alpha_i - \alpha_o} - \frac{\alpha_o}{\alpha_i - \alpha_o} \quad (13)$$

so that the average Nusselt number is simply:

$$Nu = \frac{\alpha_i - \alpha_o}{2\pi} Q. \quad (14)$$

RESULTS AND DISCUSSION

When the student groups have finally succeeded in writing a computer program that converges and with (apparently) no mistakes, the instructor must emphasize the importance of first testing the code against known results when possible. This is the case here. An excellent way to verify the accuracy of the numerical code developed in bipolar coordinates is to first consider the concentric case. Bipolar coordinates do not degenerate into cylindrical coordinates as the eccentricity ratio tends towards zero. Nevertheless, calculations can be performed for, say, $\varepsilon = 0.01$, and the results obtained should be very close to those obtained in the concentric case.

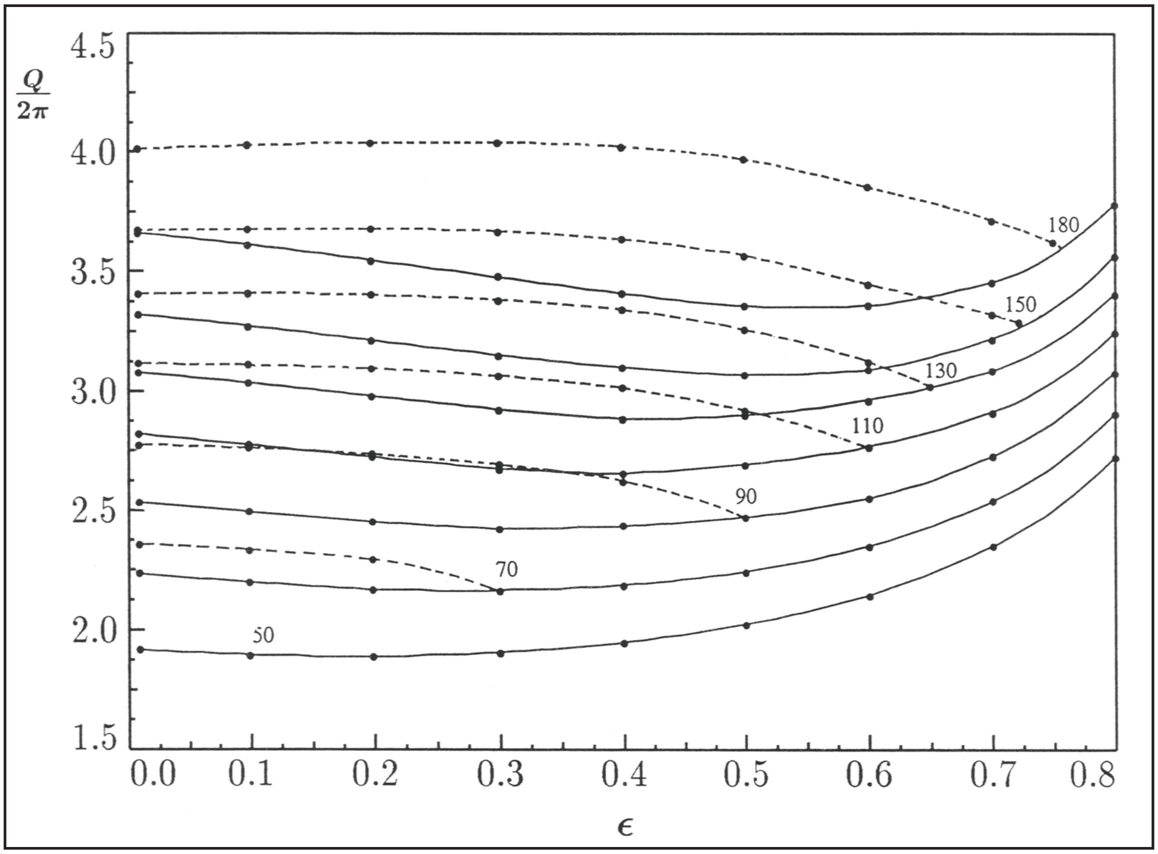

Figure 2. *Dimensionless heat flow into the eccentric annular porous medium vs. eccentricity ratio ε for different Rayleigh numbers, $r_o/r_i = 2$. The dotted curves correspond to the four-cell hydrodynamic regimes.*

As covered in Reference 1, the results for this case show that for this clearance ratio at a Rayleigh number of 65 ± 4, a second hydrodynamic regime with an extra cell can appear at the top of the layer in agreement with the hydrodynamic stability calculations of Reference 5 and the experimental investigation of Reference 6. Furthermore, the values of the Nusselt number for a given Rayleigh number agree well with those obtained previously, the difference being less than 1%.

Now that the validity of the developed numerical program in bipolar coordinates has been verified, calculations for other values of the eccentricity ratio and for different Rayleigh numbers can be performed. As ε increases, the value of the Rayleigh number at which an extra convective cell appears also increases, as expected. For example, for an eccentricity ratio of ε = 0.3, the four-cell regime appears at a Rayleigh number of 70 and for ε = 0.6, it appears at a Rayleigh number of 110 (see Figure 2). On Figure 2 we plot the heat flow loss as a function of eccentricity ratio ε for different Rayleigh numbers. Notice that for high Rayleigh numbers (for example 180, top curve on Figure 2), the minimum energy loss corresponds to a case where the eccentricity is not zero. The energy loss decrease for these Rayleigh numbers can be higher than 10%.

Finally one can show[7] that it is indeed the suppression of the extra convective cell that is responsible for the energy loss reduction. On Figure 3 we show streamlines and isotherms for Ra = 110 and $r_o/r_i = 2$ and different values of ε. At this Rayleigh number the second cell disappears for ε = 0.6.

This project was given, many years ago, to the senior/graduate students of the advanced transport phenomena course that we teach. Students worked in groups of up to four. They had one semester to write the numerical code. One group§ worked very well, their code was correct, and the graphics were excellent. We spent many hours with them running the code and analyzing data, and their work was presented at the International Heat Transfer Conference in Brighton, U.K.[8]

OTHER TUBE GEOMETRIES AND MEDIA

Horizontal tubes of elliptic cross-section can also be studied in the same way as above. Elliptical coordinates (u, v) are defined by the following transformations from Cartesian coordinates:

$$x = a \cosh u \cos v,\ y = a \sinh u \sin v,\ z = z \qquad (15)$$

where a is the focal radius, u > 0 and 0 < v < 2π. As for the bipolar coordinate system, the scale factors in the two cross-sectional directions are equal:

$$h_u = h_v = a\sqrt{\cosh^2 u - \cos^2 v}. \qquad (16)$$

§ *We have noted that almost every semester, one or more computer enthusiasts were enrolled in our course. In doing the project they don't count the hours spent in improving the code and the post treatment of results. They consider the project more as play than work. In this case two such students formed one group.*

It is interesting to note that both of these coordinate systems have been used in the past to describe the motion of planets around the sun. The dimensionless equations can be obtained in the same way as described above.

As for the eccentric cylinder system, two dimensionless parameters are required to define this geometry—the major axis ratio a_2/a_1, and the inner ellipse eccentricity b_1/a_1. Note that this coordinate system does degenerate into cylindrical coordinates as the inner cylinder eccentricity, b_1/a_1, tends toward 1. Again the accuracy of the code can be verified by considering the cylindrical case first.

The results obtained in porous media[9] are similar to those obtained with a cylindrical duct. At high enough Rayleigh numbers a secondary cell appears in the top of the layer. However, when the major axis of the ellipses are aligned with the gravity field, *i.e.*, when the transformations are:

$$x = a \sinh u \sin, \; y = a \cosh u \cos v \tag{17}$$

a curious phenomenon occurs for moderate to high eccentricities and high enough Rayleigh numbers. The secondary cell is not stable and a periodic regime establishes itself in the porous layer. We have published these results with the students who completed the project successfully.[9]

Square or rectangular ducts have also been studied and many interesting questions can be asked in the learning process. For a rectangular duct, is the heat loss the same for the cases when the long end is perpendicular or parallel to the gravity vector? For the square duct, what happens when two diagonal edges of the square are aligned with the gravity field?

All of the problems described above for a porous medium can be formulated for a fluid medium. In two dimensions, the governing equations can be solved for a vorticity/stream function formulation or using primitive variables and a SIMPLE scheme.[10]

Finally, using regime transitions in natural convection is an excellent way to introduce students to the fascinating topic of hydrodynamic stability, normally taught in graduate school.

CONCLUSIONS

Projects requiring the numerical simulation of coupled transport phenomena conservation equations are an excellent way to teach senior/graduate students many different aspects of transport phenomena and CFD. Here and in Reference 1 we have discussed natural convection in enclosed porous surfaces; different geometries have been considered. These problems are well suited as projects in a CFD course. By considering different geometries and both porous and fluid media, we have been able to give a new project each semester without difficulty. This work was a great experience both for the students and the instructor. In several instances[7-9] we were able to publish the results obtained with our undergraduate students.

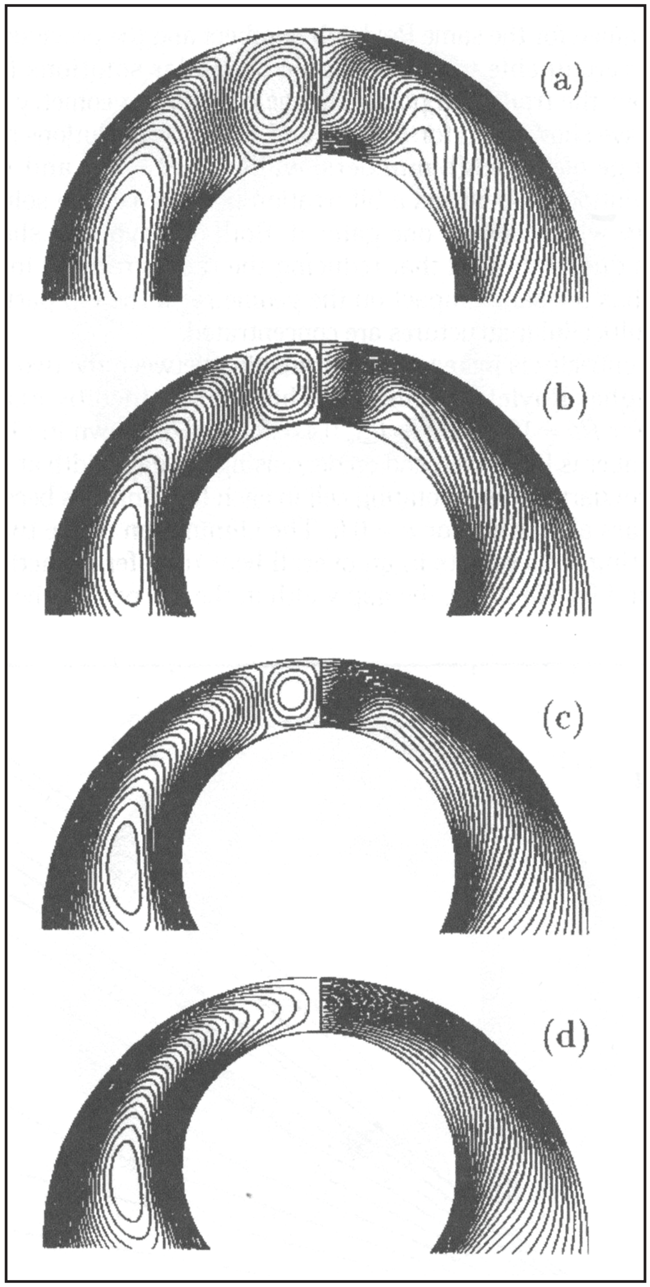

Figure 3. *Natural convection between eccentric porous cylinders. Streamlines are plotted on the left and isotherms on the right of the annulus. The parameters are: $r_o/r_i = 2$, Ra = 110, and a) ε = 0.1, b) 0.4, c) 0.5, d) 0.6. Notice that the extra cell at the top of the layer can be suppressed by a change in geometry.*

NOMENCLATURE

- a distance from origin to focal point in bipolar coordinate system, m
- **g** gravitational acceleration vector, m/s^2
- h scale factors in coordinate system, m
- r_i, r_o radius of inner and outer cylinders, m
- r position vector, m

Ra	Rayleigh number
T	absolute temperature, K
V_α, V_β	velocity components in bipolar coordinates, m/s

Greek letters

α, β	directions in bipolar coordinates
β_o	thermal expansion coefficient, K^{-1}
$\delta = r_o/r_i$	clearance ratio
ε	eccentricity ratio
Θ^+	dimensionless temperature
λ^*	thermal conductivity of the porous medium (solid and fluid)
$(\rho c_p)^*$	heat capacity per unit volume of porous medium (solid and fluid), J/m^3K
$(\rho c_p)_f$	heat capacity per unit volume of fluid, J/m^3K
ψ^+	dimensionless stream function

REFERENCES

1. Saatdjian, E., F. Lesage, and J.P.B. Mota, "Transport phenomena projects: a method to learn and to innovate. Natural convection between porous, horizontal cylinders," *Chem. Eng. Ed.*, **47**(1), 59 (2013)
2. Saatdjian, E., *Transport Phenomena: Equations and Numerical Solutions*, John Wiley and Sons, Chichester, U.K. (2000)
3. Hildebrand, F.B., *Advanced Calculus for Applications*, Prentice-Hall, Englewood Cliffs (1962)
4. Carnahan, B., H.A. Luther, and J.O. Wilkes, *Applied Numerical Methods*, John Wiley, New York, (1969)
5. Caltagirone, J.P., "Thermoconvective instabilities in a porous medium bounded by two concentric horizontal cylinders," *J. Fluid Mech.*, **76**, 337 (1976)
6. Charrier-Mojtabi, M.C., A. Mojtabi, M. Azaiez, and G. Labrosse, "Numerical and experimental study of multi-cellular free convection flows in an annular porous layer," *Int. J. Heat Mass Transfer*, **34**, 3601 (1991)
7. Barbosa Mota, J.P., and E. Saatdjian, "On the reduction of natural convection heat transfer in horizontal eccentric annuli containing saturated porous media," *Int. J. Num. Methods Heat Fluid Flow*, **7**(4), 401 (1997)
8. Barbosa Mota, J.P., J.C. Leprevost, E. Puons, and E. Saatdjian, "Natural convection in porous horizontal eccentric annuli," *IHTC-10 Heat Transfer 94*, G.F. Hewitt, editor, **5**, 435 (1994)
9. Saatdjian, E., R. Lam, A. Lefévre, and J.C. Leprevost, "Natural convection heat transfer in the annular region between porous confocal ellipses," *Numerical Methods in Fluids*, **31**(2), 513 (1999)
10. Patankar, S., *Numerical Heat Transfer*, Hemisphere, Washington, (1980) ❒