

CONSERVATION EQUATIONS FOR AXIAL FLOW IN PACKED BEDS

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INTRODUCTION

The foundational differential equations of the continuum model of the packed bed reactors cannot be solved directly.^[1] Discerning simplifying assumptions are needed to render them amenable to analysis and numerical computation. This should start with an examination of a general form of the conservation equations. Yet, over three decades ago, Rice^[2] wrote: “In teaching an introductory, graduate-level course in diffusional operations ..., I have been struck by the lack of a derivation of the complete differential equation describing dispersion effects in beds and other multiple-phase systems.” Rice derived the equation for concentration in porous beds from the first principles.

A paper by Amundson^[3] discusses in some detail the derivation of several models of reactions in packed beds. This is a scholarly article and is useful for those with experience in modelling of reactions in packed beds but does not include a step-by-step derivation of the equations. Also, a variety of equations for transient or steady state modeling of reactions is available in the literature but are specific to particular research goals without much explanation.^[4-10] But a straightforward and detailed derivation of the conservation equations for reactions in packed beds that are readily accessible by students is not yet available. Given the complicated nature of the flow in packed beds, an explicit exposition of the conservation equations could be very helpful to both graduate and undergraduate students and could serve as a unified reference for researchers for comparative discourse.

This article derives the continuity, motion and energy equations in detail for the fluid phase in axial flow in tubular packed beds in cylindrical coordinates. The derivations are made under the continuum hypothesis^[11] and the following assumptions:

1. Transport functions are radially symmetric.
2. Radial change in pressure is negligible.

3. Flow is axial in the cylindrical coordinate system.
4. The only shear stress is τ_z with only two components of τ_{zz} and τ_{rz} .
5. Only the gravitation potential energy is included.
6. Packing is uniformly distributed and is isotropic.^[12]

POROSITY AND VOID FRACTION

Figure 1 shows the schematic of a packed bed catalytic reactor filled with spherical catalyst pellets. Catalyst particles come in a variety of geometries including sphere, cylinder, or ring. The individual particles are porous inside, and within each cross section of the tube, there is the void space between the catalyst particles. Figure 2 shows the porosity and the void space in a layer of a porous bed.

Porosity and void fraction are two important concepts that are used interchangeably in the analysis of the fluid dynamics of porous media.^[13-15] The distribution of porosity in a randomly packed bed is a complicated function of axial, radial, and angular coordinates. The void fraction distribution of the bed complicates the path of the convective flux, and the distribution of the pores inside the catalyst pellets impacts the calculation of the effective diffusivity for the assessment of the diffusive flux.^[16]

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Research to unravel the distribution of porosity has been an adjunct to the study of reactions in packed beds. Several authors have demonstrated that the radial porosity distribution of mono-sized spherical particles has a Besselian type oscillatory pattern^[13,17,18] that reaches a value of 1 near the wall. Roblee et al.^[13] showed that the void fraction also has a Besselian oscillatory pattern for 1/2-in. cylinders with less radial variations for 1/2-in. Raschig rings. Not much research is available on the axial variations of porosity. Von Seckenderoff^[19] has shown that the bulk axial porosity of mono-sized spherical particles also presents an oscillatory pattern and that the oscillations are more pronounced towards the beginning and the end of the bed. From the work of Zou and Yu,^[20] they concluded that the axial variations of the bed at the extremities of the bed are negligible for bed length to particle diameter ratios of greater than 20. This is a significant conclusion because most industrial reactors belong to this category and this is perhaps the reason the axial variations of porosity are often ignored.^[21,22] Constant porosity has also been assumed to study reactions in packed beds.^[9, 23] The assumption of constant porosity has drawn some criticism. Vortmeyer and Haidegger^[22] cite constant porosity as one of the factors that fails to explain the distribution of energy in packed beds.

In this article the conservation equations are derived for isotropic packed beds with emphasis on the importance of the influence of the local void and volume fractions on the profiles of the velocity, concentration and temperature. In the interest of clarity these concepts, as we have utilized them, are explicitly defined below.

Definition 1. The local void fraction (cross sectional porosity) at a cross section z of the bed, denoted by $\varepsilon_{cp}(z, r)$, is defined as the free fractional area of a cross section of the bed normal to the flow which is available for the convective transfer of the fluid through the packing. It may be expressed as Eq. (1).¹

$$\varepsilon_{cp}(z, r) = 1 - \frac{A_{solid}(z, r)}{A_{total}} \quad (1)$$

Definition 2. The local volume fraction, denoted by $\phi(z, r)$, is defined as the free local fractional volume of the bed that is available for fluid retention.

The void fraction $\varepsilon_{cp}(z, r)$ becomes relevant when the fluid flows through a cross section of the bed. The volume fraction $\phi(z, r)$ is relevant when the variations in the accumulation within a volume element are being computed.

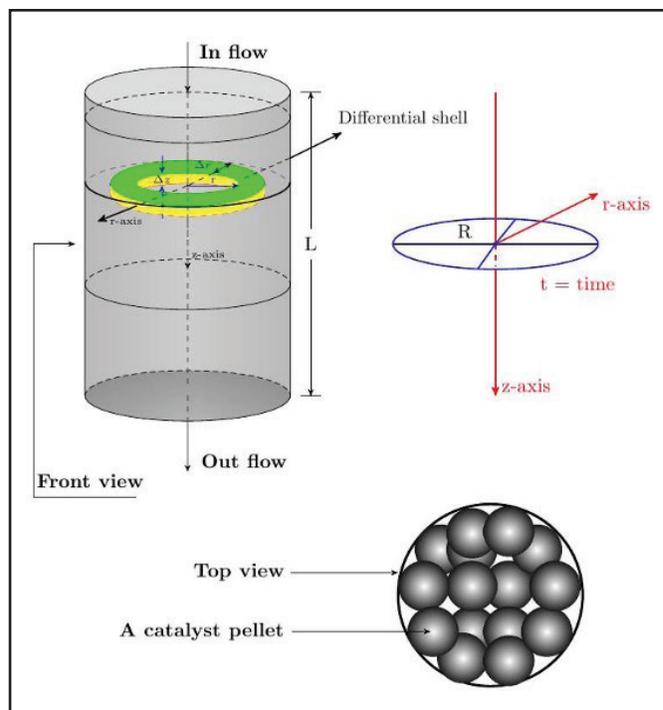


Figure 1: A schematic of a porous bed catalytic reactor.

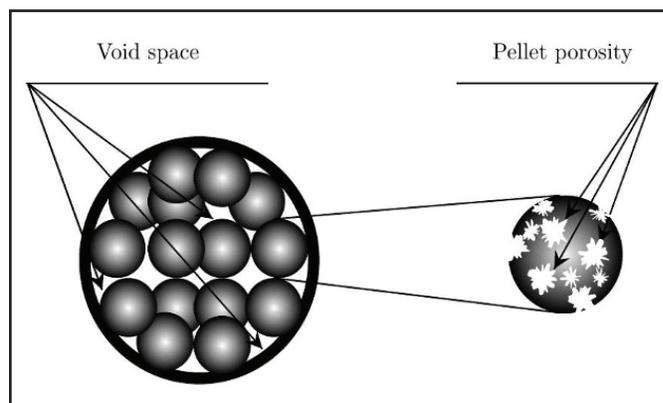


Figure 2: A view of a cross section of a porous bed.

For isotropic media, void and volume fractions are related as follows:

$$\phi(z, r) = \varepsilon_{cp}(z, r) + (1 - \varepsilon_{cp}(z, r)) \varepsilon_p$$

where ε_p is the pellet porosity. The variations of the local void fraction determine the course of the flow in a packed bed and influences the profiles of velocity, concentration and temperature.

1. Eq. (1) is a two-dimensional variation of the definition of porosity given by Mueller.^[17]

CONTINUITY

$$\{ \text{rate of accumulation of mass within the shell} \} = \{ \text{rate of mass entering the shell} - \text{rate of mass leaving the shell} \}$$

Figure 3 shows a differential cylindrical shell inside a tubular porous bed.

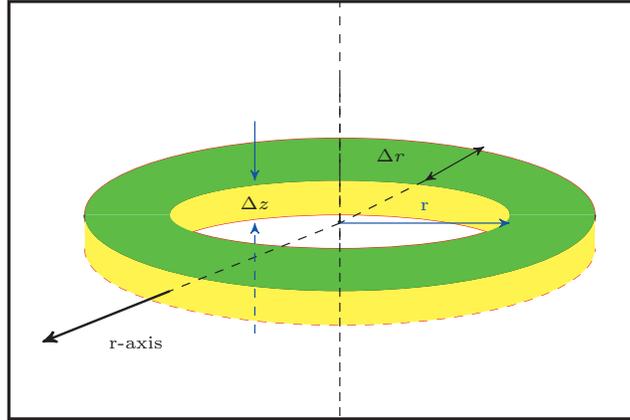


Figure 3. The geometry of a differential shell in cylindrical coordinates.

For axial flow, the material balance equation through a porous bed is shown in Eq. (2). A nomenclature list is provided at the end of this manuscript.

$$\frac{\partial}{\partial t} \left(\underbrace{2\pi r \Delta r \Delta z \phi}_{\text{volume}} \varrho \right) \approx \underbrace{2\pi r \Delta r \varepsilon_{cp}}_{\text{area}} v \varrho|_z - \underbrace{2\pi r \Delta r \varepsilon_{cp}}_{\text{area}} v \varrho|_{z+\Delta z} \quad (2)$$

If this equation is divided by $\Delta r \Delta z$ and the limit is taken as $\Delta r, \Delta z \rightarrow 0$, and the right-hand side of the equation is transposed, the continuity equation is obtained as shown in Eq. (3).

$$\phi \frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial z} (\varepsilon_{cp} v \varrho) = 0 \quad (3)$$

Dispersion

The transfer of mass for a compound A in a reaction taking place in a packed bed involves dispersion. In a (z, r) system the dispersive mass fluxes ($\text{kg}/\text{m}^2/\text{s}$) for packed beds are expressed in terms of the dispersion coefficients as shown in Eq. (4).

$$J_z = -\mathcal{D}_z \frac{\partial \varrho w_A}{\partial z}, \quad J_r = -\mathcal{D}_r \frac{\partial \varrho w_A}{\partial r} \quad (4)$$

Here, w_A is the mass fraction of compound A, and \mathcal{D}_z and \mathcal{D}_r represent the axial and transversal mass dispersion coefficients, respectively. \mathcal{D}_z and \mathcal{D}_r are phenomenological coefficients and are functions of many properties of a packed bed such as viscosity, density, particle size distribution, etc. The unit of the dispersion coefficient is m^2/s . A review of the dispersion coefficients with several correlations is given by Delgado.^[24]

A shell mass balance in units of kg/s for a compound A is given in Eq. (5).

$$\begin{aligned} \underbrace{\frac{\partial}{\partial t}(2\pi r \Delta r \Delta z \phi \rho w_A)}_{\text{rate of accumulation}} \approx & \underbrace{\frac{2\pi r \Delta r \varepsilon_{cp} v}{\text{volumetric flow rate}} \rho w_A \Big|_z}_{\text{volumetric flow rate}} - \underbrace{\frac{2\pi r \Delta r \varepsilon_{cp} v}{\text{volumetric flow rate}} \rho w_A \Big|_{z+\Delta z}}_{\text{net rate by convection}} \\ & + \underbrace{2\pi r \Delta r \varepsilon_{cp} J_z^A \Big|_z - 2\pi r \Delta r \varepsilon_{cp} J_z^A \Big|_{z+\Delta z}}_{\text{net rate by axial dispersion}} + \underbrace{2\pi r \Delta z \varepsilon_{cp} J_r^A \Big|_r - 2\pi r \Delta z \varepsilon_{cp} J_r^A \Big|_{r+\Delta r}}_{\text{net rate by radial dispersion}} \\ & + \underbrace{2\pi r \Delta r \Delta z \phi R_{wA}}_{\text{rate of consumption/production}} \end{aligned} \quad (5)$$

In Eq. (5), v is the average mass velocity, R_{wA} is the rate of production or consumption of A and has the units of kg/m³/s. If Eq. (5) is divided by $2\pi r \Delta r \Delta z$ and the limit is taken as $\Delta r, \Delta z \rightarrow 0$, Eq. (6) is obtained.

$$\frac{\partial}{\partial t}(\phi \rho w_A) = -\frac{\partial}{\partial z}(\varepsilon_{cp} v \rho w_A) - \frac{\partial}{\partial z}(\varepsilon_{cp} J_z^A) - \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} J_r^A) + \phi R_{wA} \quad (6)$$

Now, if the dispersive fluxes are replaced by the expressions in Eq. (4), with some algebraic rearrangement, Eq. (6) in the final form will be as Eq. (7).

$$\phi \frac{\partial}{\partial t}(\rho w_A) + \frac{\partial}{\partial z}(\varepsilon_{cp} v \rho w_A) = \frac{\partial}{\partial z}(\varepsilon_{cp} \mathcal{D}_z \frac{\partial \rho w_A}{\partial z}) + \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \mathcal{D}_r \frac{\partial \rho w_A}{\partial r}) + \phi R_{wA} \quad (7)$$

Often the species continuity equation is expressed in terms of the concentration of compound A. If ρw_A is replaced by $M_A C^A$, where M_A is the molecular weight of A, Eq. (8) is obtained.

$$\phi \frac{\partial}{\partial t}(C^A) + \frac{\partial}{\partial z}(\varepsilon_{cp} v^* C^A) = \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \mathcal{D}_r \frac{\partial C^A}{\partial r}) + \frac{\partial}{\partial z}(\varepsilon_{cp} \mathcal{D}_z \frac{\partial C^A}{\partial z}) + \phi R_{C^A} \quad (8)$$

In Eq. (8), v^* is the average molar velocity and R_{C^A} is the reaction rate which is expressed in the units of mol/m³/s.

Since for heterogeneous catalytic reactions the intrinsic reaction rate \mathfrak{R}^A is often expressed in the units of mol/kg_{catal.}/s, it is preferable to express the rate functions R_{wA} and R_{C^A} in terms of \mathfrak{R}^A . They are given as follows:

$$R_{wA} = \frac{\rho_p M_A}{\phi} \mathfrak{R}^A, \quad R_{C^A} = \frac{\rho_p}{\phi} \mathfrak{R}^A$$

ρ_p is the packing density and is the weight of catalyst divided by the volume of the empty reactor. In terms of the intrinsic reaction rate, the equations for compound A are as shown below. Equation (9) is given in terms of mass and Eq. (10) is in terms of concentration. Converting from mass to concentration could be a bit of work for students: hence, two equations.

$$\phi \frac{\partial}{\partial t}(\rho w_A) + \frac{\partial}{\partial z}(\varepsilon_{cp} v \rho w_A) = \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \mathcal{D}_r \frac{\partial \rho w_A}{\partial r}) + \frac{\partial}{\partial z}(\varepsilon_{cp} \mathcal{D}_z \frac{\partial \rho w_A}{\partial z}) + \rho_p M_A \mathfrak{R}^A \quad (9)$$

$$\phi \frac{\partial}{\partial t}(C^A) + \frac{\partial}{\partial z}(\varepsilon_{cp} v^* C^A) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \varepsilon_{cp} \mathcal{D}_r \frac{\partial C^A}{\partial r} \right) + \frac{\partial}{\partial z} \left(\varepsilon_{cp} \mathcal{D}_z \frac{\partial C^A}{\partial z} \right) + \rho_p \mathfrak{R}^A \quad (10)$$

Normally, several concurrent reactions each with a rate of $\mathcal{R}_i(z, r)$ contribute to the rate of A so that, in general, the following relation represents the overall reaction rate for the compound A while taking stoichiometry into account as appropriate.

$$\mathfrak{R}^A = \sum_i \mathcal{R}_i(z, r), i = 1, \dots$$

MOTION

{ rate of change of momentum of the fluid within the shell } = { rate of transfer of momentum into and out of the shell by transport of mass } + { sum of forces acting on the fluid }

The forces acting on the fluid are the viscous and body forces. The only body force considered here is gravity. In axial flow in the cylindrical coordinate system, the only shear stress is τ_z with two components of τ_{zz} and τ_{rz} .²

In a differential shell of a packed bed, the solid particles are stationary and the balance must be made on the fluid by remembering that the shell is a porous material. The z -component force balance on the shell in Figure 3 is as follows.

$$\begin{aligned} \frac{\partial}{\partial t} \underbrace{(2\pi r \Delta r \Delta z \phi \rho v)}_{\text{momentum}} &\approx \underbrace{(2\pi r \Delta r \varepsilon_{cp} v \rho)}_{\text{net force from transfer of mass}} \Big|_z - \underbrace{(2\pi r \Delta r \varepsilon_{cp} v \rho)}_{\text{net force from transfer of mass}} \Big|_{z+\Delta z} \\ &+ \underbrace{(2\pi r \Delta r \varepsilon_{cp} P)}_{\text{net force from fluid pressure}} \Big|_z - \underbrace{(2\pi r \Delta r \varepsilon_{cp} P)}_{\text{net force from fluid pressure}} \Big|_{z+\Delta z} + \underbrace{2\pi r \Delta z \varepsilon_{cp} \tau_{rz} \Big|_r - 2\pi r \Delta z \varepsilon_{cp} \tau_{rz} \Big|_{r+\Delta r}}_{\text{net force from lateral shear stress}} \\ &\quad + \underbrace{2\pi r \Delta r \varepsilon_{cp} \tau_{zz} \Big|_z - 2\pi r \Delta r \varepsilon_{cp} \tau_{zz} \Big|_{z+\Delta z}}_{\text{net force from axial shear stress}} + \underbrace{2\pi r \Delta r \Delta z \phi \rho g}_{\text{gravitational force}} \end{aligned} \quad (11)$$

If this equation is divided by $2\pi r \Delta r \Delta z$ and the limit is taken as $\Delta r, \Delta z \rightarrow 0$, it will result in Eq. (12).

$$\phi \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial z}(\varepsilon_{cp} \rho v^2) = -\frac{\partial}{\partial z}(\varepsilon_{cp} P) - \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + \phi \rho g \quad (12)$$

This equation can be expanded and grouped as Eq. (13).

$$\begin{aligned} v \left[\phi \frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial z}(\varepsilon_{cp} \rho v) \right] + \phi \rho \frac{\partial v}{\partial t} + \varepsilon_{cp} \rho v \frac{\partial v}{\partial z} = \\ -\frac{\partial}{\partial z}(\varepsilon_{cp} P) - \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + \phi \rho g \end{aligned} \quad (13)$$

By the equation of continuity, this simplifies as Eq. (14).

$$\rho \left[\phi \frac{\partial v}{\partial t} + \varepsilon_{cp} v \frac{\partial v}{\partial z} \right] = -\frac{\partial}{\partial z}(\varepsilon_{cp} P) - \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + \phi \rho g \quad (14)$$

2. For a description of shear stresses see Analysis of Heat and Mass Transfer by Eckert. ^[25]

ENERGY

Eq. (15) represents an approximate balance of energy for the shell in Figure 3. The shell energy balance is expressed with respect to the internal and kinetic energies. The entries in Eq. (15) are all in the unit of J/s.

$$\begin{aligned}
 & \underbrace{\frac{\partial}{\partial t} \left[2\pi r \Delta r \Delta z \phi \left(\rho U + \frac{1}{2} \rho v^2 \right) \right]}_{\text{net rate of change in energy}} \approx \\
 & \underbrace{\frac{2\pi r \Delta r \varepsilon_{cp} v}{\text{volumetric flow rate}} \left(\rho U + \frac{1}{2} \rho v^2 \right) \Big|_z - \frac{2\pi r \Delta r \varepsilon_{cp} v}{\text{volumetric flow rate}} \left(\rho U + \frac{1}{2} \rho v^2 \right) \Big|_{z+\Delta z}}_{\text{net rate of energy transported axially}} \\
 & + \underbrace{\frac{2\pi r \Delta z \varepsilon_{cp}}{\text{open } r\text{-cross sectional area for fluid}} q_r \Big|_r - \frac{2\pi r \Delta z \varepsilon_{cp}}{\text{open } r\text{-cross sectional area for fluid}} q_r \Big|_{r+\Delta r}}_{\text{net rate of energy conducted radially by the fluid}} \\
 & + \underbrace{\frac{2\pi r \Delta r \varepsilon_{cp}}{\text{open } z\text{-cross sectional area for fluid}} q_z \Big|_z - \frac{2\pi r \Delta r \varepsilon_{cp}}{\text{open } z\text{-cross sectional area for fluid}} q_z \Big|_{z+\Delta z}}_{\text{net rate of energy conducted axially by the fluid}} \\
 & + \underbrace{(2\pi r \Delta r \varepsilon_{cp} P) v \Big|_z - (2\pi r \Delta r \varepsilon_{cp} P) v \Big|_{z+\Delta z}}_{\text{net rate of work done on the shell by static pressure}} \\
 & + \underbrace{\frac{2\pi r \Delta r \varepsilon_{cp} v \tau_{zz} \Big|_z - 2\pi r \Delta r \varepsilon_{cp} v \tau_{zz} \Big|_{z+\Delta z}}{\text{net rate of work against } z\text{-component of surface force}} + \frac{2\pi r \Delta z \varepsilon_{cp} v \tau_{rz} \Big|_r - 2\pi r \Delta z \varepsilon_{cp} v \tau_{rz} \Big|_{r+\Delta r}}{\text{net rate of work against } r\text{-component of surface force}}} \\
 & + \underbrace{\frac{2\pi r \Delta r \Delta z \phi \rho g v}{\text{rate of work done against gravity}} + \frac{2\pi r \Delta r \Delta z \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i)}{\text{net rate of enthalpy to or from reactions}} + \frac{2\pi r \Delta r \Delta z \phi E_e}{\text{rate of external energy}}}
 \end{aligned} \tag{15}$$

If we divide Eq. (15) by $2\pi r \Delta z \Delta r$ we will obtain Eq. (16).

$$\begin{aligned}
 \phi \frac{\partial}{\partial t} \left(\rho U + \frac{1}{2} \rho v^2 \right) & \approx \frac{\varepsilon_{cp} v \left(\rho U + \frac{1}{2} \rho v^2 \right) \Big|_z - \varepsilon_{cp} v \left(\rho U + \frac{1}{2} \rho v^2 \right) \Big|_{z+\Delta z}}{\Delta z} \\
 + \frac{1}{r} \frac{r \varepsilon_{cp} q_r \Big|_r - r \varepsilon_{cp} q_r \Big|_{r+\Delta r}}{\Delta r} + \frac{\varepsilon_{cp} q_z \Big|_z - \varepsilon_{cp} q_z \Big|_{z+\Delta z}}{\Delta z} + \frac{\varepsilon_{cp} P v \Big|_z - \varepsilon_{cp} P v \Big|_{z+\Delta z}}{\Delta z} \\
 + \frac{\varepsilon_{cp} v \tau_{zz} \Big|_z - \varepsilon_{cp} v \tau_{zz} \Big|_{z+\Delta z}}{\Delta z} + \frac{1}{r} \frac{r \varepsilon_{cp} v \tau_{rz} \Big|_r - r \varepsilon_{cp} v \tau_{rz} \Big|_{r+\Delta r}}{\Delta r} + \phi \rho g v \\
 + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e
 \end{aligned} \tag{16}$$

Now, all terms are in the unit of J/m³/s. If the limit is taken as $\Delta z, \Delta r \rightarrow 0$, we obtain Eq. (17).

$$\begin{aligned}
 \phi \frac{\partial}{\partial t} \left[\rho \left(U + \frac{1}{2} v^2 \right) \right] & = - \frac{\partial}{\partial z} \left[\varepsilon_{cp} v \rho \left(U + \frac{1}{2} v^2 \right) \right] - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) \\
 & - \frac{\partial}{\partial z} (\varepsilon_{cp} v \tau_{zz}) - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} v \tau_{rz}) - \frac{\partial}{\partial z} (\varepsilon_{cp} P v) + \phi \rho g v \\
 & + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e
 \end{aligned} \tag{17}$$

To simplify, Bird et al.^[26] subtracted the kinetic energy equation from the energy equation. This is done here too. We subtract the following equation,

$$\phi \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) = - \frac{\partial}{\partial z} \left(\frac{1}{2} \varepsilon_{cp} \rho v^2 \right) - v \frac{\partial}{\partial z} (\varepsilon_{cp} P) - v \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z} (\varepsilon_{cp} \tau_{zz}) + \phi \rho g v,$$

which is Eq. (A-9) in the Appendix, from Eq. (17) to obtain Eq. (18).

$$\begin{aligned} \phi \frac{\partial}{\partial t} (\rho U) = - \frac{\partial}{\partial z} (\varepsilon_{cp} v \rho U) & - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ & - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (18)$$

If we expand the derivative on the left-hand side and the first derivative term on the right-hand side and transpose them, we will obtain Eq. (19).

$$\begin{aligned} U \phi \frac{\partial \rho}{\partial t} + \rho \phi \frac{\partial U}{\partial t} + U \frac{\partial}{\partial z} (\varepsilon_{cp} v \rho) + \varepsilon_{cp} v \rho \frac{\partial}{\partial z} (U) = - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (19)$$

Upon regrouping and factoring, we will get Eq. (20).

$$\begin{aligned} U \left[\phi \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} (\varepsilon_{cp} v \rho) \right] + \rho \phi \frac{\partial U}{\partial t} + \varepsilon_{cp} v \rho \frac{\partial U}{\partial z} = \\ - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (20)$$

By the equation of continuity Eq. (20) simplifies as follows:

$$\begin{aligned} \rho \phi \frac{\partial U}{\partial t} + \varepsilon_{cp} v \rho \frac{\partial U}{\partial z} = - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (21)$$

Energy Equation in Terms of C_V

Since $dU = \left[T \left(\frac{\partial P}{\partial T} \right)_v - P \right] dV + C_v dT$, the partial derivatives of the internal energy are given by Eqs. (22) and (23).

$$\frac{\partial U}{\partial z} = \left[T \left(\frac{\partial P}{\partial T} \right)_v - P \right] \frac{\partial V}{\partial z} + C_v \frac{\partial T}{\partial z} \quad (22)$$

$$\frac{\partial U}{\partial t} = \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] \frac{\partial V}{\partial t} + C_V \frac{\partial T}{\partial t} \quad (23)$$

If we substitute Eqs. (22) and (23) in Eq. (21) we obtain the following equation:

$$\begin{aligned} \rho \phi C_V \frac{\partial T}{\partial t} + \rho \phi \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] \frac{\partial V}{\partial t} + \varepsilon_{cp} v \rho C_V \frac{\partial T}{\partial z} + \varepsilon_{cp} v \rho \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] \frac{\partial V}{\partial z} = \\ - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} \\ + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (24)$$

If the terms on the left-hand side are regrouped and V is replaced by $1/\rho$, we obtain Eq. (25).

$$\begin{aligned} \rho C_V \left[\phi \frac{\partial T}{\partial t} + \varepsilon_{cp} v \frac{\partial T}{\partial z} \right] - \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] \frac{1}{\rho} \left[\phi \frac{\partial \rho}{\partial t} + \varepsilon_{cp} v \frac{\partial \rho}{\partial z} \right] = - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) \\ - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (25)$$

We define the substantial derivative operator for porous beds as in Eq. (26),

$$\frac{D^p}{Dt} = \phi \frac{\partial}{\partial t} + \varepsilon_{cp} v \frac{\partial}{\partial z} \quad (26)$$

and express Eq. (25) as Eq. (27).

$$\begin{aligned} \rho C_V \frac{D^p T}{Dt} - \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] \frac{1}{\rho} \frac{D^p \rho}{Dt} = - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (27)$$

Since $\frac{1}{\rho} \frac{D^p \rho}{Dt} = \frac{D^p \ln \rho}{Dt}$, the energy equation may be expressed as Eq. (28).

$$\begin{aligned} \rho C_V \frac{D^p T}{Dt} - \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] \frac{D^p \ln \rho}{Dt} = \\ - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} \\ + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (28)$$

If Fourier's law of heat conduction is used, the energy equation in terms of temperature is obtained as shown in Eq. (29).

$$\begin{aligned} \rho C_v \frac{D^p T}{Dt} - \left[T \left(\frac{\partial P}{\partial T} \right)_v - P \right] \frac{D^p \ln \rho}{Dt} = \\ \frac{1}{r} \frac{\partial}{\partial r} \left(r \varepsilon_{cp} k_{r, \text{eff}} \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left(\varepsilon_{cp} k_{z, \text{eff}} \frac{\partial T}{\partial z} \right) - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (29)$$

Energy Equation in Terms of C_p

The energy equation in terms of C_p is more commonly used in engineering practice. Since $dU = dH - dPV$, in terms of the fundamental thermodynamic relations, $\frac{dU}{dt}$ and $\frac{dU}{dz}$ may be expressed as follows:

$$\begin{aligned} \frac{\partial U}{\partial t} &= -T \left(\frac{\partial V}{\partial T} \right)_p \frac{\partial P}{\partial t} + C_p \frac{\partial T}{\partial t} - P \frac{\partial V}{\partial t} \\ \frac{\partial U}{\partial z} &= -T \left(\frac{\partial V}{\partial T} \right)_p \frac{\partial P}{\partial z} + C_p \frac{\partial T}{\partial z} - P \frac{\partial V}{\partial z} \end{aligned}$$

If these relations are substituted in Eq. (21) we will obtain Eq. (30).

$$\begin{aligned} -\phi \rho T \left(\frac{\partial V}{\partial T} \right)_p \frac{\partial P}{\partial t} + \phi \rho C_p \frac{\partial T}{\partial t} - \phi \rho P \frac{\partial V}{\partial t} - \varepsilon_{cp} \rho v T \left(\frac{\partial V}{\partial T} \right)_p \frac{\partial P}{\partial z} \\ + \varepsilon_{cp} \rho v C_p \frac{\partial T}{\partial z} - \varepsilon_{cp} \rho v P \frac{\partial V}{\partial z} = -\frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) \\ - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (30)$$

If rearranged, Eq. (30) will be as Eq. (31)

$$\begin{aligned} \rho_p \left[\phi \frac{\partial T}{\partial t} + \varepsilon_{cp} v \frac{\partial T}{\partial z} \right] - \rho T \left(\frac{\partial V}{\partial T} \right)_p \left[\phi \frac{\partial P}{\partial t} + \varepsilon_{cp} v \frac{\partial P}{\partial z} \right] - \rho P \left[\phi \frac{\partial V}{\partial t} + \varepsilon_{cp} v \frac{\partial V}{\partial z} \right] = \\ -\frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} \\ + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (31)$$

By using the substantial derivative and letting $v = \frac{1}{\rho}$, we will get Eq. (32).

$$\begin{aligned} \rho C_p \frac{D^p T}{Dt} + \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{D^p P}{Dt} + \frac{P}{\rho} \left[\phi \frac{\partial \rho}{\partial t} + \varepsilon_{cp} v \frac{\partial \rho}{\partial z} \right] = \\ -\frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} \\ + \phi \rho_p \sum_i \mathcal{R}_i(-\Delta H_i) + \phi E_e \end{aligned} \quad (32)$$

Since

$$\phi \frac{\partial \rho}{\partial t} + \varepsilon_{cp} v \frac{\partial \rho}{\partial z} = \left[\phi \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} (\varepsilon_{cp} \rho v) \right] - \rho \frac{\partial \varepsilon_{cp} v}{\partial z} = -\rho \frac{\partial \varepsilon_{cp} v}{\partial z} \quad (33)$$

by the continuity equation, Eq. (32) is reduced to Eq. (34).

$$\begin{aligned} \rho C_p \frac{D^p T}{Dt} + \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{D^p P}{Dt} - P \frac{\partial \varepsilon_{cp} v}{\partial z} = & -\frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) \\ & - \varepsilon_{cp} P \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (34)$$

If we transpose $-P \frac{\partial \varepsilon_{cp} v}{\partial z}$ we will obtain Eq. (35).

$$\begin{aligned} \rho C_p \frac{D^p T}{Dt} + \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{D^p P}{Dt} = & -\frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) + P \frac{\partial \varepsilon_{cp} v}{\partial z} - \varepsilon_{cp} P \frac{\partial v}{\partial z} \\ & - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (35)$$

Since $P \frac{\partial \varepsilon_{cp} v}{\partial z} - \varepsilon_{cp} P \frac{\partial v}{\partial z} = P v \frac{\partial \varepsilon_{cp}}{\partial z}$, the final form of the energy equation in terms of C_p is given by Eq. (36).

$$\begin{aligned} \rho C_p \frac{D^p T}{Dt} + \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{D^p P}{Dt} = & -\frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} q_r) - \frac{\partial}{\partial z} (\varepsilon_{cp} q_z) \\ & + P v \frac{\partial \varepsilon_{cp}}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (36)$$

With Fourier's law, Eq. (36) will be as Eq. (37).

$$\begin{aligned} \rho C_p \frac{D^p T}{Dt} + \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{D^p P}{Dt} = & \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon_{cp} k_{r, \text{eff}} \frac{\partial T}{\partial r}) + \frac{\partial}{\partial z} (\varepsilon_{cp} k_{z, \text{eff}} \frac{\partial T}{\partial z}) \\ & + P v \frac{\partial \varepsilon_{cp}}{\partial z} - \varepsilon_{cp} \tau_{zz} \frac{\partial v}{\partial z} - \varepsilon_{cp} \tau_{rz} \frac{\partial v}{\partial r} + \phi \rho_p \sum_i \mathcal{R}_i (-\Delta H_i) + \phi E_e \end{aligned} \quad (37)$$

DISCUSSION

Note the term $P v \frac{\partial \varepsilon_{cp}}{\partial z}$ in Eq. (37). This term is present only because of the porosity of the bed. If $\varepsilon_{cp}(z, r) = \phi(z, r) = 1$, Eq. (37) corresponds to the energy equation for an empty pipe and it reduces to equation (11.2 - 5) for axial flow in *Transport Phenomena*, only missing the enthalpy of the reactions and external source of energy.

The dynamics of the flow field of the catalytic reactions in packed beds are complicated. The void fraction is the distinctive indicator of this complexity and is a function that determines the course of the flow in the interior of the bed.

Assumptions concerning the void and volume fractions directly influence the outcomes of the resolution of conservation equations of the processes in porous media.

The transfer of mass and energy to and from the stationary and the mobile phases makes the direct analysis of the reaction systems in porous beds virtually impossible. Alternative pseudo-heterogeneous and heterogeneous models^[8,10,27,28] distinguish the flow of mass or energy in the mobile phase from that in the interior of the particle, and their representations are manifest with different sets of differential equations. The interphase interactions are included by the laws of heat and mass transfer. Even then the partial differential

equations of these models cannot be readily solved because of the complexity and the indefinability of the boundary conditions that are inherent in the structure of the packing.^[29,30]

To resolve packed-bed equations, pseudo-continuum models are constructed from heterogeneous and homogeneous models by making assumptions on the transport parameters of the dispersion of mass, energy and porosity. Pseudo-homogeneous models do not directly account for the particles in the bed^[31,32] and doubt has been raised about their efficacy for a realistic assessment of reactions in packed beds.^[33] Concerning the heterogeneous models, even simplifying assumptions are hard to come by for the local resolution of dispersion and energy equations.

This brief analysis should demonstrate the importance of having access to an explicit and detailed model of fluid dynamics in packed beds because the nature of the flow in a porous bed is obscure and simplifying assumptions need to begin from an analysis of the fundamental and descriptive equations. It is hoped that these derivations serve as an easy reference for both students and researchers who are interested in studying the characteristics of heterogeneous reactions in packed beds.

NOMENCLATURE

A tube cross sectional area, m^2
 C^A concentration of compound A , mol/m^3
 C_p constant pressure heat capacity, $J/kg/K$
 C_v constant volume heat capacity, $J/kg/K$
 E_e externally supplied or removed energy, $J/s/m^3$
 g gravitational acceleration, m/s^2
 H_i reaction molar enthalpy, J/mol
 J_r radial mass flux, $kg/m^2/s$
 J_z axial mass flux, $kg/m^2/s$
 $k_{r,eff}$ effective radial thermal conductivity of fluid, $W/m/K$
 $k_{z,eff}$ effective axial thermal conductivity of fluid, $W/m/K$
 L length of reactor, m
 M_A molecular weight, kg/mol
 P pressure, Pascal
 $q_r = q_r(z, r, t)$, radial conductive heat flux in the fluid phase, W/m^2
 $q_z = q_z(z, r, t)$, radial conductive heat flux in the fluid phase, W/m^2
 R internal radius of reactor, m
 r radial coordinate, m
 R_C^A reaction rate, $mol/m^3/s$
 R_W^A reaction rate $kg/m^3/s$
 T temperature, K

t time, s
 U internal energy per unit mass, J/kg
 V specific volume, m^3/kg
 $v = v(z, r, t)$, average mass velocity, m/s
 $v^* = v^*(z, r, t)$, average molar velocity, m/s
 W_A mass fraction
 z axial coordinate, m
 \mathcal{D}_r radial dispersion coefficient, m^2/s
 \mathcal{D}_z axial dispersion coefficient, m^2/s
 $\mathfrak{R}^A = \mathfrak{R}^A(z, r, t)$, intrinsic reaction rate of compound A , $mol/kg_{catal.}/s$
 $\mathcal{R}_i = \mathcal{R}_i(z, r, t)$, intrinsic reaction rate of compound A , $mol/kg_{catal.}/s$
 $\phi = \phi(z, r)$, volume fraction
 τ_{rz} r component of shear stress, N/m^2
 τ_{zz} z component of shear stress, N/m^2
 ϵ_p pellet porosity
 $\epsilon_{cp} = \epsilon_{cp}(z, r)$, cross sectional porosity (void fraction)
 $\rho = \rho(z, r, t)$, fluid density, kg/m^3
 ρ_p packing density per empty reactor volume, $kg_{catal.}/m^3$

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APPENDIX: A MECHANICAL ENERGY BALANCE

The equation of motion, Eq. (14), is listed as Eq. (A-1)

$$\phi \rho \frac{\partial v}{\partial t} + \varepsilon_{cp} \rho v \frac{\partial v}{\partial z} = -\frac{\partial}{\partial z}(\varepsilon_{cp} P) - \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + \phi \rho g \quad (\text{A-1})$$

We adopt the method proposed by Bird et al.^[8] and multiply Eq. (A-1) by v to obtain Eq.

$$\phi \rho v \frac{\partial v}{\partial t} + \varepsilon_{cp} \rho v^2 \frac{\partial v}{\partial z} = -v \frac{\partial}{\partial z}(\varepsilon_{cp} P) - v \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + v \phi \rho g \quad (\text{A-2})$$

Equation (A-2) may be written as Eq. (A-3).

$$\phi \rho \frac{\partial}{\partial t} \left(\frac{1}{2} v^2 \right) + \varepsilon_{cp} \rho v \frac{\partial}{\partial z} \left(\frac{1}{2} v^2 \right) = -v \frac{\partial}{\partial z}(\varepsilon_{cp} P) - v \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + v \phi \rho g \quad (\text{A-3})$$

Note the following derivatives obtained from the product rule.

$$\rho \frac{\partial}{\partial t} \left(\frac{1}{2} v^2 \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) - \frac{1}{2} v^2 \frac{\partial}{\partial t}(\rho) \quad (\text{A-4})$$

$$\varepsilon_{cp} \rho v \frac{\partial}{\partial z} \left(\frac{1}{2} v^2 \right) = \frac{\partial}{\partial z} \left(\frac{1}{2} \varepsilon_{cp} \rho v^2 v \right) - \frac{1}{2} v^2 \frac{\partial}{\partial z}(\varepsilon_{cp} \rho v) \quad (\text{A-5})$$

If we substitute Eqs. (A-4) and (A-5) into Eq. (A-3) we will have Eq. (A-6)

$$\begin{aligned} \phi \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) - \frac{1}{2} v^2 \frac{\partial}{\partial t}(\rho) \right] + \frac{\partial}{\partial z} \left(\frac{1}{2} \varepsilon_{cp} \rho v^2 v \right) - \frac{1}{2} v^2 \frac{\partial}{\partial z}(\varepsilon_{cp} \rho v) = -v \frac{\partial}{\partial z}(\varepsilon_{cp} P) \\ - v \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + v \phi \rho g \end{aligned} \quad (\text{A-6})$$

If rearranged, Eq. (A-7) will be as follows:

$$\begin{aligned} \phi \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) + \frac{\partial}{\partial z} \left(\frac{1}{2} \varepsilon_{cp} \rho v^2 v \right) - \frac{1}{2} v^2 \left[\phi \frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial z}(\varepsilon_{cp} \rho v) \right] = \\ -v \frac{\partial}{\partial z}(\varepsilon_{cp} P) - v \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + v \phi \rho g \end{aligned} \quad (\text{A-7})$$

By the continuity equation Eq. (3), this equation reduces to Eq. (A-8).

$$\phi \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) + \frac{\partial}{\partial z} \left(\frac{1}{2} \varepsilon_{cp} \rho v^2 v \right) = -v \frac{\partial}{\partial z}(\varepsilon_{cp} P) - v \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + v \phi \rho g \quad (\text{A-8})$$

If the second term on the left-hand side is transposed, Eq. (A-9) will be obtained.

$$\phi \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) = -\frac{\partial}{\partial z} \left(\frac{1}{2} \varepsilon_{cp} \rho v^2 v \right) - v \frac{\partial}{\partial z}(\varepsilon_{cp} P) - v \frac{1}{r} \frac{\partial}{\partial r}(r \varepsilon_{cp} \tau_{rz}) - v \frac{\partial}{\partial z}(\varepsilon_{cp} \tau_{zz}) + v \phi \rho g \quad (\text{A-9})$$