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ANALOGY BETWEEN HEAT EXCHANGER AND PACKED COLUMN CALCULATIONS

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INTRODUCTION

The use of analogies in chemical engineering education helps students who may have a diverse science and engineering background visualize complex concepts and learn intuitively.[1,2] It offers a structural framework where students can link new and abstract topics to something that is more comprehensible, thus enhancing their learning experience and contributing to their fullest potential. Analogies are emphasized in the transport phenomena.[3,4] Examples include the thermal-electrical analogy of conductive heat transfer and Chilton and Colburn J-factor analogy relating friction factors, heat transfer coefficients, and mass transfer coefficients.

In heat exchanger calculations the effectiveness-number of transfer units (NTU) method is very useful if the two outlet temperatures are to be solved for a given exchanger.[4] It avoids the tedious trial-and-error procedure involved in the log mean temperature difference method. This paper presents an analogy between heat exchanger and packed column calculations that may aid student learning in transport and separation principles. Using this method, the outlet compositions of gas and liquid streams for a given packed column can be determined without the use of iterations. Even though the NTU is used interchangeably with N_{OG} (number of overall gas transfer units) to calculate the height of a packed column,[5,6] it is shown in this work that its definition must be modified in order to derive the same effectiveness-NTU equation. It is noted that the effectiveness-NTU has been applied to study mass transfer in countercurrent absorbers;[7] however, the nomenclature does not always follow that used in an exchanger. To the best knowledge of the author, this is the first unambiguous presentation of such an analogy between the two unit operations.

ANALOGY BETWEEN PACKED COLUMN AND EXCHANGER

A schematic of a packed absorption column is shown in Figure 1. The gas and liquid streams flow in a countercurrent configuration. x and y represent the liquid and gas compositions, respectively. Subscripts i and o represent inlet and outlet, respectively.

The following assumptions are made in the model development:

1. The operation is at steady state.
2. Both gas and liquid streams are dilute. Their flow rates are constant along the column.
3. The gas-liquid equilibrium is a straight line.
4. The mass transfer coefficient is constant along the column.

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The rate of absorption in the control volume shown in Figure 1 can be written as:

$$-Gdy = -Ldx = K_y a A_c (y - y^*) dz \qquad (1)$$

where K_y is the overall mass transfer coefficient based on the gas phase with a mole ratio driving force, a is the interfacial area per unit volume between gas and liquid, and A_c is the cross-sectional area of the column. y^* is the gas composition that is in equilibrium with x:

$$y^* = mx \qquad (2)$$

where m is the slope of the equilibrium curve.

Following the equivalent resistance approach commonly used in heat transfer through a composite wall, Eq. (1) can be written as:

$$-\frac{dy}{\frac{1}{G}} = -\frac{mdx}{\frac{m}{L}} = -\frac{d(y - y^*)}{\frac{1}{G} - \frac{m}{L}} = K_y a A_c (y - y^*) dz \qquad (3)$$

Treating $(y - y^*)$ as a single variable, an integration of Eq. (3) from the bottom (b) to the top (t) of the column yields:

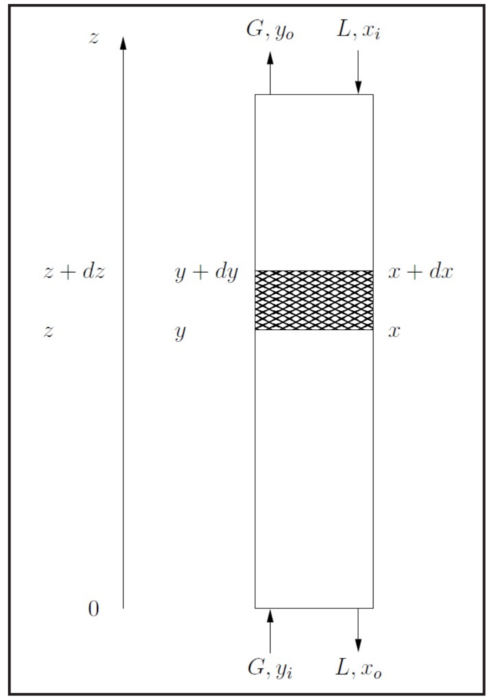

***Figure 1**. Schematic of a packed column.*

$$H = \int_b^t dz = \int_b^t -\frac{1}{K_y a A_c \left(\frac{1}{G} - \frac{m}{L}\right)} \frac{d(y - y^*)}{(y - y^*)} = -\frac{1}{K_y a A_c \left(\frac{1}{G} - \frac{m}{L}\right)} \ln \frac{(y - y^*)_t}{(y - y^*)_b} \qquad (4)$$

where H is the height of the column.

Note that Eq. (3) also implies:

$$-\int_b^t \frac{dy}{\frac{1}{G}} = -\int_b^t \frac{d(y - y^*)}{\frac{1}{G} - \frac{m}{L}} \qquad (5)$$

or

$$G\left(\frac{1}{G} - \frac{m}{L}\right) = \frac{(y - y^*)_b - (y - y^*)_t}{y_i - y_o} \qquad (6)$$

Therefore,

$$H = -\frac{G}{K_y a A_c \frac{(y - y^*)_b - (y - y^*)_t}{y_i - y_o}} \ln \frac{(y - y^*)_t}{(y - y^*)_b} = \frac{G}{K_y a A_c} \frac{y_i - y_o}{(y - y^*)_{lm}} \qquad (7)$$

where the log mean (lm) driving force is:

$$(y - y^*)_{lm} = \frac{(y - y^*)_t - (y - y^*)_b}{\ln \frac{(y - y^*)_t}{(y - y^*)_b}} \qquad (8)$$

As a result, the packed column height can be described by:

$$H = N_{OG} H_{OG} \qquad (9a)$$

$$N_{OG} = \frac{y_i - y_o}{(y - y^*)_{lm}} \tag{9b}$$

$$H_{OG} = \frac{G}{K_y a A_c} \tag{9c}$$

where N_{OG} is number of overall gas transfer units and H_{OG} is the height of one overall gas transfer unit.

Eq. (9b) is much shorter than Colburn's equation:[8]

$$N_{OG} = \frac{1}{1 - \frac{mG}{L}} \ln\left[\left(1 - \frac{mG}{L}\right)\frac{y_i - y_o^*}{y_o - y_o^*} + \frac{mG}{L}\right] \tag{10}$$

Based on Eq. (7), the total mass transfer rate N is:

$$N = K_y a A_c H (y - y^*)_{lm} \tag{11}$$

An analogy between packed columns and heat exchangers can be clearly seen in Eqs. (12) and (13) below:

$$N = G(y_i - y_o) = \frac{L}{m}(y_o^* - y_i^*) = K_y a A_c H (y - y^*)_{lm} \tag{12}$$

and

$$q = C_h(T_{hi} - T_{ho}) = C_c(T_{co} - T_{ci}) = UA(T_h - T_c)_{lm} \tag{13}$$

where q is heat transfer rate in an exchanger, C is the product of mass flow rate and heat capacity, T is temperature, and U is overall heat transfer coefficient. Subscripts h and c represent hot and cold streams, respectively.

Following the same methodology used in the effectiveness-NTU method for an exchanger, the theoretical maximum mass transfer rate (N_{max}) where the column is infinitely high is defined as:

$$N_{max} = \min\{G, L/m\}\,(y_i - y_i^*) \tag{14}$$

The effectiveness (ϵ) is defined as the ratio between the actual mass transfer rate (N) and the theoretical maximum mass transfer rate (N_{max}). Or,

$$N = \epsilon \min\{G, L/m\}\,(y_i - y_i^*) \tag{15}$$

Moreover, the NTU in a packed column is defined as:

$$NTU = \frac{K_y a A_c H}{\min\{G, L/m\}} \tag{16}$$

Note that Eqs. (11) and (15) imply:

$$\epsilon \min\{G, L/m\}\,(y_i - y_i^*) = K_y a A_c H (y - y^*)_{lm} \tag{17}$$

A combination of Eqs. (16) and (17) yields:

$$\epsilon(y_i - y_i^*) = NTU(y - y^*)_{lm} \tag{18}$$

or,

$$\ln\frac{y_i - y_o^*}{y_o - y_i^*} = NTU\,\frac{(y_i - y_o^*) - (y_o - y_i^*)}{\epsilon(y_i - y_i^*)} \tag{19}$$

The left-hand side of Eq. (19) can be written as:

$$\ln\frac{y_i - y_i^* + y_i^* - y_o^*}{y_o - y_i + y_i - y_i^*} = \ln\frac{1-(y_o^* - y_i^*)/(y_i - y_i^*)}{1-(y_i - y_o)/(y_i - y_i^*)} = \ln\frac{1-\epsilon\min\{G, L/m\}/(L/m)}{1-\epsilon\min\{G, L/m\}/G} \tag{20}$$

The right-hand side of Eq. (19) can be written as:

$$NTU\frac{(y_i - y_o) - (y_o^* - y_i^*)}{\epsilon(y_i - y_i^*)} = NTU\frac{N/G - N/(L/m)}{N/\min\{G, L/m\}} \tag{21}$$

Therefore, Eq. (19) is converted to:

$$\ln\frac{1-\epsilon\min\{G, L/m\}/(L/m)}{1-\epsilon\min\{G, L/m\}/G} = NTU\frac{1/G - 1/(L/m)}{1/\min\{G, L/m\}} \tag{22}$$

which can be further simplified as:

$$\ln\frac{1-\epsilon C_r}{1-\epsilon} = NTU(1-C_r), \text{if } G < L/m \tag{23a}$$

$$\ln\frac{1-\epsilon}{1-\epsilon C_r} = NTU(C_r - 1), \text{if } G > L/m \tag{23b}$$

where the capacitance ratio (C_r) is defined as:

$$C_r = \frac{\min\{G, L/m\}}{\max\{G, L/m\}} \tag{24}$$

Eqs. (23a) and (23b) are essentially the same, regardless of the relative values of G and L/m. Starting from either Eq. (23a) or (23b), ϵ may be solved as:

$$\epsilon = \frac{1-\exp\big(-NTU(1-C_r)\big)}{1-C_r\exp\big(-NTU(1-C_r)\big)} \tag{25}$$

If $C_R = 1$, Eq. (25) can be simplified using L'Hôpital's rule:

$$\epsilon = \lim_{C_r\to 1}\frac{1-\exp\big(-NTU(1-C_r)\big)}{1-C_r\exp\big(-NTU(1-C_r)\big)} = \frac{NTU}{NTU+1} \tag{26}$$

Eqs. (25) and (26) are consistent with those in a countercurrent heat exchanger.[4] In the special case where $C_r = 0$, Eq. (25) becomes:

$$\epsilon = 1 - \exp(-NTU) \tag{27}$$

or

$$NTU = -\ln(1-\epsilon) \tag{28}$$

For a heat exchanger, $C_r = 0$ means one stream undergoes a phase change. For a packed column, $C_r = 0$ if the equilibrium line is much lower than the operating line, or $y^* << y$. Such an assumption is usually made in limestone scrubber design for SO_2 removal.[9] In this case, the mass transfer rate is:

$$N = \epsilon\min\{G, L/m\}(y_i - y_i^*) = \epsilon G y_i \tag{29}$$

Therefore, ϵ may be interpreted as the removal efficiency of the absorbate.

When $y^* << y$ is valid, Eq. (9b) can be simplified as follows:

$$N_{OG} = \frac{y_i - y_o}{(y-y^*)_{lm}} = \frac{y_i - y_o}{y_{lm}} = -\ln\left(\frac{y_o}{y_i}\right) = -\ln(1-\epsilon) \tag{30}$$

which is consistent with Eq. (28).

The analogy between heat exchanger and packed column is summarized in Table 1. It is worth pointing out that N_{OG} is equivalent to NTU only when $G < L/m$. If $G > L/m$, from Eq. (16), $NTU = \frac{K_y a A_c H}{L/m} = \frac{K_y a A_c H}{G} / \frac{L/m}{G} = N_{OG}/C_r$. This is a major difference from previous work.[7] The effectiveness-NTU method is handy when the column height is specified while the two outlet compositions are to be determined.

TABLE 1
Analogy Between Heat Exchanger and Packed Column

	Heat Exchanger	Packed Column
Transfer Rate	q	N
Capacitance 1	C_h	G
Capacitance 2	C_c	L/m
Process Variable 1	T_h	y
Process Variable 2	T_c	y^*
Maximum Transfer Rate	$q_{max} = \min\{C_h, C_c\}\,(T_{hi} - T_{ci})$	$N_{max} = \min\{G, L/m\}\,(y_i - y_i^*)$
Log Mean Driving Force	$(T_h - T_c)_{lm}$	$(y - y^*)_{lm}$
Overall Conductance	UA	$K_y a A_c H$
Number of Transfer Units	$UA/\min\{C_h, C_c\}$ $\frac{(T_{hi}-T_{ho})}{\Delta T_{lm}}$ if $C_h < C_c$ $\frac{(T_{co}-T_{ci})}{\Delta T_{lm}}$ if $C_h > C_c$	$K_y a A_c H/\min\{G, L/m\}$ $\frac{(y_i-y_o)}{(y-y^*)_{lm}} = N_{OG}$ if $G < L/m$ $\frac{(y_o^*-y_i^*)}{(y-y^*)_{lm}} = \frac{N_{OG}}{C_r}$ if $G > L/m$
Capacitance Ratio	$C_r = \min\{C_h, C_c\}/\max\{C_h, C_c\}$	$C_r = \min\{G, L/m\}/\max\{G, L/m\}$

ILLUSTRATIVE EXAMPLES

Example 1: Acetone in air is being absorbed by water in a packed tower having a cross-sectional area of 0.186 m^2 at 293 K and 1 atm. At these conditions, the equilibrium relation is given by $y^* = mx$, where $m = 1.186$. The inlet air contains 2.6 mol % acetone and 0.5 % on outlet. The air flow is 13.65 kmol/hr, and the pure water inlet flow is 45.36 kmol/hr. Film coefficients for the given flow in the tower are $\overline{k_y a} = 3.8\times10^{-2}$ kmol/(s·m^3) and $\overline{k_x a} = 6.2\times10^{-2}$ kmol/(s·m^3). Determine the tower height. Assume dilute streams. This problem is taken from the Geankoplis textbook.[4]

A summary of the solution is listed below. Detailed calculations are provided in a MATLAB® code in Table 2.

$$K_y a = \frac{1}{\frac{1}{k_y a} + \frac{m}{k_x a}} = 0.022\ \mathrm{kmol/(m^3 \cdot s)}$$

$$H_{OG} = \frac{G}{K_y a A_c} = 0.926\ \mathrm{m}$$

$$N_{OG} = \frac{y_i - y_o}{(y - y^*)_{lm}} = 2.035$$

$$N_{OG} = \frac{1}{1 - \frac{mG}{L}} \ln\left[\left(1 - \frac{mG}{L}\right)\frac{y_i - y_o^*}{y_o - y_o^*} + \frac{mG}{L}\right] = 2.035$$

$$H = n_{OG} H_{OG} = 1.89\ \mathrm{m}$$

As one can see, Eq. (9b) and Eq. (10) yield the same N_{OG}.

Example 2: The column height is given as 1.89 m. The inlet flow rates and compositions are the same as Example 1. The same $K_y a$ and A_c can also be used. Calculate the outlet compositions x_O and y_O.

The effectiveness-NTU method is used to avoid iterations. A summary of the solution is listed below. Detailed calculations are provided in a MATLAB code in Table 2.

$$NTU = \frac{K_y a A_c H}{\min\{G, L/m\}} = 2.035$$

$$\epsilon = \frac{1 - \exp\big(-NTU(1 - C_r)\big)}{1 - C_r \exp\big(-NTU(1 - C_r)\big)} = 0.808$$

$$N = \epsilon \min\{G, L/m\} \left(y_i - m x_i\right) = 0.287 \text{ kmol/hr}$$

Therefore,

$$y_o = y_i - N/G = 0.005$$

$$x_o = x_i + N/L = 0.0063$$

The outlet compositions match with those in Example 1.

Example 3: The local environmental protection agency mandates that your company increase the SO_2 collection efficiency of your limestone scrubber from its current level, 90%, to 96%. You propose to do this by increasing the height of your scrubber. By what factor must the height be increased?

Using Eq. (28),

$$\frac{H_2}{H_1} = \frac{NTU_2}{NTU_1} = \frac{\ln(1 - \epsilon_2)}{\ln(1 - \epsilon_1)} = \frac{\ln(1 - 0.96)}{\ln(1 - 0.90)} = 1.398$$

Therefore, the height of the scrubber must be increased by 40%. The relationship between collection efficiency and column height is highly nonlinear! A MATLAB code for the detailed calculations is listed in Table 2.

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TABLE 2
MATLAB Code for the Illustrative Examples

```
clear all;clc;
%% calculate column height if three inlet/outlet compositions are known
(example 1)
m = 1.186;  %slope of y* = mx
Kya = 1/(1/0.038+m/0.062); %Kya in kmol/(m3 sec)
Ac = 0.186; %cross-sectional area in m2
G = 13.65;  %gas flow in kmol/hr
L = 45.36;  %liquid flow in kmol/hr
yi = 0.026; %inlet gas composition
yo = 0.005; %outlet gas composition
xi = 0;     %inlet liquid composition
xo = G*(yi-yo)/L+xi; %outlet liquid composition
y_ys1 = yi-m*xo;% y-y* at bottom
y_ys2 = yo-m*xi;% y-y* at top
delta_lm = (y_ys1 - y_ys2)/log(y_ys1/y_ys2);% log mean of y-y*
NoG = (yi-yo)/delta_lm; % calculate NoG using log mean
NoG_2 = 1/(1-m*G/L)*log((1-m*G/L)*yi/yo+m*G/L); %NoG using Colburn's
Equation
HoG = G/3600/Kya/Ac; % HoG in meter
H = NoG*HoG; % column height in meter
fprintf('NoG = %7.3f, NoG_2 = %7.3f, HoG (m) = %7.3f, H (m) =
%7.3f\n',NoG,NoG_2,HoG, H);

%% Calculate yo and xo if column height is known (example 2)
NTU = Kya*Ac*H/min(G/3600,L/m/3600); % NTU
Cr = min(G,L/m)/max(G,L/m);% Cr
epsilon = (1-exp(-NTU*(1-Cr)))/(1-Cr*exp(-NTU*(1-Cr))); % effectiveness
Nmax = min(G,L/m)*(yi-m*xi); % max mass transfer rate in kmol/hr
N = epsilon*Nmax; % actual mass transfer rate in kmol/hr
yo = yi - N/G; % y at outlet
xo = xi + N/L; % x at outlet
fprintf('NTU = %7.3f, epsilon = %7.3f, N (kmol/hr) = %7.3f, yo = %7.4f, xo =
%7.4f\n',NTU,epsilon,N,yo,xo);

%% Calculate height ratio (example 3)
epsilon1 = 0.90;
epsilon2 = 0.96;
H2_H1 = log(1-epsilon2)/log(1-epsilon1);
fprintf('H2/H1 = %7.3f\n',H2_H1);
```

```
NoG =    2.035, NoG_2 =    2.035, HoG (m) =    0.926, H (m) =    1.885
NTU =    2.035, epsilon =    0.808, N (kmol/hr) =    0.287, yo =  0.0050, xo =
0.0063
H2/H1 =    1.398
```