A Unified Approach

# **(eJ n a classroom**

# **TURBULENT TRANSFER PROCESSES**

G. D. FULFORD\* and D. C.T.PEI *University of Waterloo Waterloo, Ontario, Canada* 

**In** studying heat and mass transfer **in** flowing fluid systems it is often helpful to use a unified approach to stress the analogies and similarities which exist among the transfers of heat, mass and momentum. Although transfer processes in turbulent fluid streams are of great practical importance, only the simplest cases can be dealt with theoretically at present by means of the statistical approach to the study of turbulence, and even in these cases, the unified attack has not been used systematically. Often the topic of turbulence is treated as though it affects only the fluid-mechanical aspects of a problem, and the concomitant heat and mass transfer effects, which are of particular chemical engineering interest, are not stressed.

In this note we will attempt to show by considering a simple case that the unified approach can be readily and conveniently applied to the study of turbulent transfer processes. A general equation is derived describing the dynamic propagation behavior of the dimensionless Eulerian double correlation parameter for the turbulent fluctuating parts of a transferred intensive property at two neighboring points in an incompressible isotropic turbulent fluid field. From this, it is shown that the well-known von Kármán-Howarth and Corrsin equations fall out directly as special cases when the intensive property is assumed to be momentum, heat energy, or mass of a component per unit volume of the system: Although no new results are obtained, and although turbulent transfer processes are rarely encountered under isotropic conditions in practice, it is felt that the generalization is of interest from a pedagogical viewpoint as an illustration of the unified approach, which obviously can also be applied usefully in the more "practical" phenomenological approach to the study of turbulent transfer processes.

\*Present Address: Hope, Spur Tree P.O., Jamaica, West Indies.

Relative to a stationary Cartesian coordinate system, the general equation expressing the conservation of an intensive property of instantaneous concentration  $P$  (i.e.,  $P$  is the quantity of property per unit volume of system at any instant, and has a time-averaged part  $\bar{p}$  and a turbulent fluctuating part P') may be expressed for a turbulent fluid as (1) :

> $\frac{\partial P}{\partial t} + \frac{5}{3^{\frac{5}{2}}} \frac{\partial}{\partial x_{\dot 1}} (v_{\dot 3} P) = - \ \frac{5}{3^{\frac{5}{2}}} \frac{\partial}{\partial x_{\dot 3}} (\stackrel{\ \ \ \eta}{\mathcal{M}}_3) - \frac{5}{3^{\frac{5}{2}}} \frac{\partial}{\partial x_{\dot 3}} (F_{\dot 3}) + G$  $(1)$

If the time-averaged values of the property concentration  $\bar{p}$ , the molecular flux of the property  $\bar{\pi}$  the velocity vector  $\bar{v}$ , and the rates of generation of the property at the surfaces and within the bulk of an element of the fluid,  $\bar{F}$ and  $\bar{g}$  are assumed to be zero, so that isotropic fluctuations relative to a zero base are considered, as is customary, the conservation equation becomes:  $\epsilon \in \mathbb{R}^2 \setminus \xi$ 

$$
\frac{\partial F'}{\partial t} + j\frac{3}{27} \frac{\partial}{\partial x_j} (v'_j P') = -j\frac{3}{27} \frac{\partial}{\partial x_j} (\pi'_j) - j\frac{3}{27} \frac{\partial}{\partial x_j} (F'_j) + G'
$$
 (2)

If secondary molecular transfer processes (such as those due to the Dufour and Soret effects) are ignored  $(*)$ , the flux  $\pi'$  can be written in terms of the gradient of the concentration P' and a generalized isotropic molecular kinematic transfer property M (assumed constant):

$$
\mathcal{N}_{j} = - M \frac{\partial}{\partial x_{s}} (\mathbf{P}^{\prime})
$$
\n(3)

and equation (2) becomes:

$$
\frac{\partial p'}{\partial t} + j^{\frac{7}{2}} 1 \frac{\partial}{\partial x_j} (v'_j P') = M \frac{3}{3^{\frac{7}{2}}} 1 \frac{\partial^2 p'}{\partial x_j^2} - j^{\frac{7}{2}} 1 \frac{\partial}{\partial x_j} (F'_j) + G' \qquad (4)
$$

<sup>\*</sup> Note that this restriction is not very clearly stated in the usual treatments of turbulent transfer processes.



George D. Fulford is Senior Project Engineer with Alumina Partners of Jamaica. George grew up in Jamaica and earned the BSc and PhD ('62) at University of Birmingham, England. His experience includes several years with DuPont in photo products and three years teaching at University of Waterloo, Ontario with special interest in the area of transfer processes. He is a member of AIChE and IChE, London and is a registered professional engineer in Ontario. (left photo)

David C. T. Pei obtained his education at McGill University, finishing in 1961. He is a member of AIChE and CSChE and is currently serving as Associate Chairman of the Chemical Engineering Department at Waterloo. His teaching interests include Fundamentals and Applications of Momentum, Heat and Mass Transport Processes. (right photo)

Equation  $(4)$  is now written for a point A in the fluid by placing a subscript A on the various quantities, and the resulting equation is multiplied throughout by the value of the fluctuating concentration  $P'_B$  at the same instant at a point B distant  $\mathfrak X$  from A; equation (4) is also written for point B(subscript B on quantities) and multiplied throughout by the fluctuating concentration  $P'_{A}$  at A \*\*. Bearing in mind that  $P'_{B}$  is not a function of  $x_A$ , nor is  $P'_A$  a function of  $x_B$ , though both are functions of time, we obtain

$$
P'_{B} \frac{\partial P'_{A}}{\partial t} + j\frac{\bar{2}}{21} \frac{\partial}{\partial x_{jA}} (v'_{jA} P'_{A} P'_{B}) = M \frac{\bar{2}}{3\bar{z}_{1}} \frac{\partial^{2} (P'_{A} P'_{B})}{\partial x_{jA}^{2}} - j\frac{\bar{2}}{z_{1}} \frac{\partial (F'_{A} P'_{B})}{\partial x_{jA}} + P'_{B} G'_{A}
$$
(5)  

$$
P'_{A} \frac{\partial P'_{B}}{\partial t} + j\frac{\bar{2}}{z_{1}} \frac{\partial}{\partial x_{jB}} (v'_{jB} P'_{B} P'_{A}) = M \frac{\bar{2}}{z_{1}} \frac{\partial^{2} (P'_{B} P'_{A})}{\partial x_{jA}^{2}} - j\frac{\bar{2}}{z_{1}} \frac{\partial (F'_{A} P'_{A})}{\partial x_{jB}} + P'_{A} G'_{B}
$$
(6)

The distance vector  $r$  between the points A and B is now written as

with 
$$
\Sigma = \delta_1 + \delta_3 + \delta_k
$$
 (7)

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Hence,

$$
\frac{3}{4} \cdot \frac{3}{4} = -\frac{3}{8x_{1A}} = \frac{3}{8x_{1B}} \cdot \frac{3}{8x_{1}^{2}} = \frac{3}{8x_{1A}^{2}} = \frac{3^{2}}{8x_{1B}^{2}} \cdot \frac{3^{2}}{8x_{1B}^{2}}
$$
 (8)

Using these definitions, equations (5) and (6) are added and the sum is then time-averaged to give

$$
\frac{\partial}{\partial t} \frac{(\overline{P_4 P_3})}{(\overline{P_4 P_3})} + \frac{\frac{3}{2}}{3\overline{\epsilon}_1} \frac{\partial}{\partial \delta_3} (\overline{\overline{v_3} P_3 P_4} - \overline{v_3 P_4 P_5}) = 2M \frac{\frac{3}{2}}{3\overline{\epsilon}_1} \frac{\partial^2}{\partial \delta_3^2} (\overline{\overline{P_4 P_2}}) +
$$
\n(1) (II) (II) (III) (III) (III) (1) (IV) (V)





(\*) Simplifications have been made by invoking the properties of homogeneous isotropic pulsations:  $P''_{1A} = P''_{1A} = P''_A = P''_A = P''$ , etc., where the double prime denotes the r.m.s. value of the corresponding fluctuating quantity.

(\*\*) Note that in the momentum transfer case the fluctuating part of the static pressure p is used by convention.

The first, second (double) and third type dimensionless Eulerian spatial correlation coefficients  $c^I$ ,  $c^I$ ,  $c^{II}$  between fluctuations of quantities at points A and B at the same instant in time are now defined in general form as shown by equations  $(10)$  -  $(15)$  in Table 1. In terms of these correlation coefficients, equation (9) can then be written:

(i) for P as a scalar quantity:

$$
\frac{\partial \sigma^{II}}{\partial t} + (\nu'') \frac{3}{4} \frac{1}{4} \frac{3}{4} \frac{1}{4} \frac{1}{4}
$$

(ii) *tor* P as a vector quantity:

oCII 3 III III 3 a2cII \_:\_::\_\_liti + (v'?ji;1 ll (c ( - c ( )) a 2H J.f1 3t .:....::.U + a6°: i,jk A,B) ij,k A,B 062 j <sup>J</sup> 3 ---- - - -- - - 1 - jt1 ..\_},,(FjBpi A- FJAPkB) + - 1 \_ (~-~) (P'?2 oo. (P '?2 (17)

These general equations can then be rewritten simply for the commonest cases of transport in a turbulent field, when P represents the momentum, heat, or mass of component X, per unit volume of multicomponent incompressible fluid.

<sup>\*\*</sup> When P is a vector quantity, the i-component of the equation (4) at A is multiplied by  $P'_{kB}$  and the kcomponent of the equation at B is multiplied by  $P'_{iA}$ .

Table 2. Equivalents of terms in general equation for cases of Heat, Mass and Momentum<br>transfer in an incompressible isotropic turbulent field.



 $\sp( ^+)$ Constant p (incompressibility) is assumed throughout. The additional complexities arising when the fluid is compressible  $(p = \bar{p} + p')$  can be readily appreciated at this point.

- $($ <sup>++</sup>) It is assumed that the heats of mixing and homogeneous chemical reaction<br>can be neglected in comparison with other terms.
- $($ <sup>+++</sup>) A first-order chemical reaction (homogeneous) of component X is assumed<br>Simple relationships are possible only for this case and the case of no<br>reaction. The problem is discussed in detail by Corrsin  $\binom{8}{2}$ .
- The fluxes are defined here as the quantities of momentum, heat or mass<br>transferred per unit time per unit area normal to the transfer direction<br>by molecular mechanisms relative to the mass-average velocity of the  $(*)$ system.
- $(**)$ It is assumed that the viscous dissipation of flow energy to heat is zero. This is customary, but is never exactly justified.
- There is no mechanism by which mass of component  $X$  may be generated at a surface in a fluid phase. Generation of  $X$  at a catalyst surface increase, for instance (surface of a fluid phase) by heterogeneous reactions mus  $(***)$ equations.

The corresponding equivalents of each term in the general equations  $(16)$ ,  $(17)$  are given in Table 2. Substituting these equivalents, we obtain the propagation equations for  $c<sup>\pi</sup>$  in the cases of Momentum transfer:

$$
\frac{\delta}{\delta t} \binom{M_{0,11}^{-1}}{13} + (\mathbf{v}'') \frac{3}{3} \frac{1}{13} \frac{\delta}{\delta \delta_{j}} \binom{M_{0,11}^{-1}}{13} \binom{M_{0,11}^{-1}}{13} - \frac{M_{0,111}^{-1}}{13} \binom{M_{0,11}^{-1}}{13} = \frac{2\mu}{\beta} \frac{3}{3} \frac{1}{13} \frac{\delta^{2}}{13} \binom{M_{0,11}^{-1}}{23} \tag{18}
$$

Heat transfer:

$$
\frac{\partial}{\partial t} \left( {}^{T}C^{II} \right) + (\nu \gamma) \frac{3}{4} \frac{1}{4} \frac{\partial}{\partial \delta} \left( {}^{T}C^{III}_{iB} - {}^{T}C^{III}_{iA} \right) = 2 \alpha \frac{3}{4} \frac{1}{4} \frac{3}{4} \frac{2}{\delta \delta_{i}^{2}} \left( {}^{T}C^{II} \right)
$$
(19)

Mass transfer:

$$
\frac{\partial}{\partial t} \left( {^m\!c}^{\Pi} \right) \; + \; \left( \nu \,{}^{\prime} \right) \, \underset{J=1}{\overset{7}{\rightarrow}} \, \frac{\partial}{\partial \delta} \left( {^m\!c}^{\Pi \Pi}_{i \, B} \; - \; {^m\!c}^{\Pi \Pi}_{i \, A} \right) \; = \; 2 D_{XY} \; \underset{J=1}{\overset{7}{\rightarrow}} \; \frac{\partial^2}{\partial \delta^2} \left( {^m\!c}^{\Pi \Pi} \right) \; - \; 2 \kappa_1^* \left( {^m\!c}^{\Pi \Pi} \right) \quad \text{(20)}
$$

As can be seen, the equations describing the behavior of  $c^{\pi}$  also involve the next higher correlation,  $c^m$ , as a result of the closure problem.

## ... we show that the unified approach can be readily and conveniently applied to the study of turbulent transfer processes.

Up to this point, no use has been made of the assumed isotropic nature of the turbulent field except to somewhat simplify the definitions made in Table 1. The isotropic properties of the turbulence can now be invoked to represent  $c^{\pi}$  and  $c<sup>m</sup>$  in terms of scalar functions f, h, q, w of the time t and the distance r between measurement points which have the appropriate transformation properties. Using the usual manipulations  $*(2, 3, 4, 5)$ , which need not be repeated here, we finally obtain the von Kármán - Howarth equation (6) for the dynamic behavior of the Eulerian double velocity correlation:

$$
\frac{\partial}{\partial t} \left[ (v^{\prime\prime})^2 f \right] = 2 \mathcal{V} (v^{\prime\prime})^2 \left( \frac{\partial}{\partial r} + \frac{\mu}{r} \right) \left( \frac{\partial f}{\partial r} - \frac{v^{\prime\prime} h}{r} \right) \tag{21}
$$

the Corrsin equation (7) for the dynamic behavior of the Eulerian double temperature correlation:

$$
\frac{\partial}{\partial t}\Big[(\mathbb{T}'')^2w_p\Big] = 2\alpha\big(\mathbb{T}'')^2\Big[\frac{\partial^2 w_p}{\partial r^2} + \frac{2}{r}\frac{\partial w_p}{\partial r}\Big] = 2(\mathbb{T}'')^2v'\Big(\frac{\partial q_p}{\partial r} + \frac{2q_p}{r}\Big) \qquad (22)
$$

and the Corrsin equation (8) for the dynamic behavior of the double concentration correlation (for the case involving a first-order chemical reaction):

$$
\frac{\partial}{\partial t} \Big[ (\rho^{\nu \nu}_X)^2 w_m \Big] \quad = \; 2 D_{\chi Y} (\rho^{\nu \nu}_X)^2 \; \left[ \frac{\partial^2 w_m}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial w_m}{\partial r^2} \right] \; + \; 2 (\rho^{\nu \nu}_X)^2 v^{\nu \nu} \frac{\partial q_m}{\partial r} + \; \frac{2 q_m}{r} \; ) \; - \; 2 k_1^{\ast} (\rho^{\nu \nu}_X)^2 w_m \qquad \text{(23)}
$$

With suitable assumptions as to the behavior of the third-order terms  $(h, q)$ , these equations have been solved for particular cases to obtain the decay of the respective double correlations under isotropic turbulent conditions (5). These results are of some practical interest since the double autocorrelation terms (for the special case when  $r = 0$  are closely related at a given level of turbulence to the Reynolds stresses, and analo-

gous heat and mass transfer terms, which appear in the phenomenological studies of turbulence.

To sum up, we feel that the unified approach used here underlines the similarity between the turbulent transport of heat, mass and momentum even in so esoteric an application as the one considered here. The main sources of difference also become clear, such as the fact that in momentum transfer the quantity transferred is a vector quantity, while in heat and mass transfer, the

 $\mathcal{L}^{\#}$ It is readily shown that all the first-type correlation parameters<br>reduce to zero for isotropic turbulence.

<sup>\*</sup> An additional operation of contraction must be carried out in the case of equation (18), where P is a vector quantity.



property is a scalar, leading to slightly different forms of the main equation. The appearance of chemical reaction term in the mass transfer case is also of interest. It can also be seen that the generalized equation will make it relatively simple to obtain equations for the dynamic behavior

under the conditions considered here of other turbulently pulsating conserved quantities, such as electric charge per unit volume, which may become important in the study of turbulent plasmas.  $\Box$ 

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### **BOOK REVIEW** (from p. 127)

summary of the principal characteristics of macromolecular systems, followed by the three major sections of the book, dealing with polymer synthesis, solid state properties, and polymer rheology. The author has managed to organize and unify the main features of polymer science quite satisfactorily. The transition from subject to subject is smooth, and the informal style and sense of awareness of the students' background should make the book eminently readable and useful as an introductory text. The introductory section, the section on polymer physics, large portions of the section on polymer systhesis, and the chapter on linear viscoelasticity of polymer solids are especially well done.

The coverage is by no means comprehensive, however. It omits such important subjects as polymer solutions, molecular characterization, the chemistry and statistics of crosslinking, and effects of molecular structure on flow properties. Indeed, the weakest part of the book is its treatment of rheology and polymer processing. Also, the discussions of the glass transition, ionic polymerizations of all kinds, crystallization kinetics. and the quantitative techniques for characterizing crystalline polymers are rather cursory. Some telescoping is necessary for the reasons discussed

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