The object of this column is to enhance our readers' collection of interesting and novel problems in Chemical Engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class or in a new light or that can be assigned as a novel home problem are requested as well as those that are more traditional in nature that elucidate difficult concepts. Please submit them to Professor H. Scot Fogler, ChE Department, University of Michigan, Ann Arbor, MI 48109.

MELTING ICE CUBES PROBLEM

JUDE T. SOMMERFELD AND
PETER A. MINDERMAN
Georgia Institute of Technology
Atlanta, GA 30332

A certain chemical reaction is being carried out in the aqueous phase in a 2000-gal. reactor. The reactor is half-full and its contents are maintained at the reaction temperature of 100°F.

At a certain point in the reaction, it is desired to quench the reaction as rapidly as possible. For this purpose, the rapid dumping of 2000 lbs. of 1-inch ice cubes (at 32°F) into the reactor contents has been proposed. Calculate 1) how long it will take for all of the ice to melt and 2) what the final solution temperature will be at this time.

Prior laboratory experiments under similar conditions of agitation have shown that the overall coefficient for heat transfer at the ice surface is 200 BTU/hr.–ft^2–°F. The latent heat of fusion of ice is 144 BTU/lb., while its density is 57.5 lbs./ft^3. The physical properties of the aqueous solution may be assumed to be the same as those for water. A sketch of this system is presented in Fig. 1.

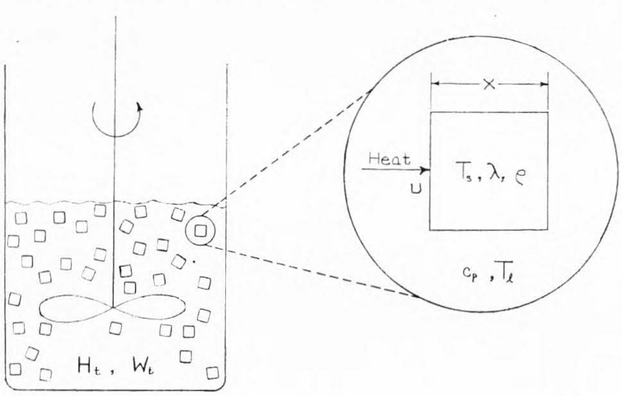
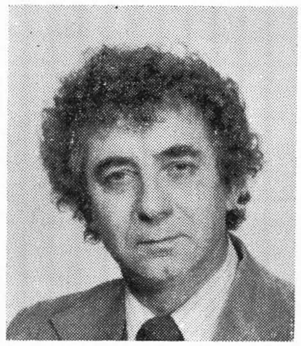
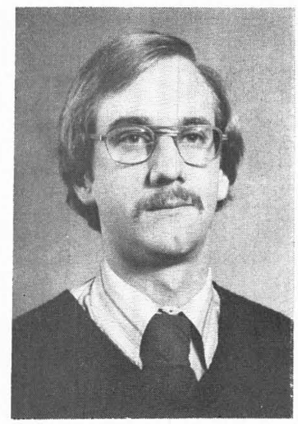


FIGURE 1. Sketch of the physical system in the melting ice cubes problem.

Jude T. Sommerfeld has been a professor of ChE at Georgia Tech since 1970. He teaches courses on process control, distillation, reactor design and process design, and his research interests include energy conservation. He has also served as a consultant to numerous industrial organizations. Prior to 1970 he had eight years of engineering and management experience with the Monsanto Company and BASF-Wyandotte Corp. Dr. Sommerfeld received his B.Ch.E. degree from the University of Detroit, and his M.S.E. and Ph.D. degrees in chemical engineering from the University of Michigan. **(L)**

Peter A. Minderman Jr. received his B.S. from Georgia Tech in 1979 and MS from the same institute in 1981. His prior industrial experience includes work with DuPont and Tennessee Eastman Co., a division of Eastman Kodak. He has recently returned to Tennessee Eastman and currently works in the Process Systems Engineering group. **(R)**

SOLUTION

The basis of the solution consists of an enthalpy balance around the solid (ice) phase. The liquid phase at the melting temperature of the solid phase is chosen as the enthalpy reference state. With this choice, the specific enthalpy (BTU/lb) of the liquid phase is equal to $c_p(T-T_s)$. Similarly, the specific enthalpy of the solid phase becomes the negative of the latent heat of fusion ($-\lambda$). Essential to the foregoing statement is the assumption that the temperature of the solid phase is always at its melting point. The rate of heat input (BTU/hr) to the solid phase is then

$$UA\ (T - T_s)$$

while the rate of accumulation is

$$-\lambda \frac{dW_s}{dt}$$

There is no heat output from the solid phase as well as no generation or consumption terms. Hence, the differential enthalpy balance equation becomes

$$UA\ (T - T_s) = -\lambda \frac{dW_s}{dt} \qquad (1)$$

It is assumed that the ice cubes are truly cubical in shape and further that they remain cubical as they melt. That is, there exists no preferential melting at any one face of the cube. Assuming that all of the cubes behave identically, the total number of cubes then also remains constant as long as ice is present. It then follows from these assumptions that

$$W_s = \rho N x^3$$

and

$$A = 6Nx^2$$

The result of substituting these equalities into Eq. (1) is

$$\frac{dx}{dt} = \frac{-2U}{\rho\lambda}(T - T_s) \qquad (2)$$

It remains then to relate the temperature of the solution (T) to the size of the melting ice cubes (x). This is achieved by performing material and enthalpy balances around the complete system. The material balance is

$$W_s + W_l = W^o_s + W^o_l = W_t \ (= \text{const})$$

or

$$W_l = W_t - W_s \qquad (3)$$

And assuming adiabatic operation, the enthalpy balance is

$$H_s + H_l = H^o_s + H^o_l = H_t \ (= \text{const})$$

or

$$H_l = H_t - H_s$$

Now

$$H^o_l = W^o_l\, c_p\, (T_o - T_s)$$
$$H^o_s = -\lambda W^o_s$$
$$H_l = W_l\, c_p\, (T - T_s)$$
$$H_s = -\lambda W_s$$

Hence

$$H_t = W^o_l\, c_p\, (T_o - T_s) - \lambda W^o_s$$

and

$$W_l\, c_p\, (T - T_s) = H_t + \lambda W_s \qquad (4)$$

Substitution of Eq. (3) for W_l into Eq. (4) and rearrangement of the result leads to

$$T - T_s = \frac{H_t + \lambda W_s}{c_p\,(W_t - W_s)} \qquad (5)$$

Finally, substitution of the above result into Eq. (2) yields

$$\frac{dx}{dt} = \frac{-2U}{\rho\lambda}\left[\frac{H_t + \lambda W_s}{c_p\,(W_t - W_s)}\right]$$

or

$$\frac{dx}{dt} = \frac{-2U}{\rho c_p}\left[\frac{(H_t/\lambda\rho N) + x^3}{(W_t/\rho N) - x^3}\right] \qquad (6)$$

Before proceeding to the integration of Eq. (6), it is convenient to make the following definitions

$$\alpha = \frac{2U}{\rho c_p}$$

$$\beta^3 = \frac{H_t}{\lambda\rho N}$$

$$\gamma^3 = \frac{W_t}{\rho N}$$

whence

$$\frac{dx}{dt} = -\alpha\left[\frac{\beta^3 + x^3}{\gamma^3 - x^3}\right] \qquad (7)$$

If one now defines a new dependent variable as

$$y = \frac{x}{\beta}$$

the result upon insertion into Eq. (7) is

$$\frac{dy}{dt} = -\frac{\alpha}{\beta}\left[\frac{1 + y^3}{\gamma^3/\beta^3 - y^3}\right]$$

And finally with the following additional definitions

$$c = \frac{\alpha}{\beta}$$

$$a = \frac{\gamma^3}{\beta^3}$$

one has

$$\frac{dy}{dt} = -c\left[\frac{1 + y^3}{a - y^3}\right] \qquad (8)$$

The integration of Eq. (8) is given in any comprehensive table of integrals. The result is

$$ct = \left[y - \frac{(a+1)}{3}\left\{\frac{1}{2}\ln\frac{(1+y)^2}{1 - y + y^2} + \sqrt{3}\ \mathrm{TAN}^{-1}\left(\frac{2y-1}{\sqrt{3}}\right)\right\}\right]_{y_o}^{y} \qquad (9)$$

where

$$y_o = \frac{x_o}{\beta}$$

For the particular case wherein the upper limit of integration is zero (that is, complete melting of the solid phase), Eq. (9) assumes the following form

$$ct\ (y = 0) = \frac{a+1}{3}\left[\frac{1}{2}\ln\frac{(1+y_o)^2}{1-y_o+y_o^2} + \sqrt{3}\ \mathrm{TAN}^{-1}\left(\frac{\sqrt{3}\ y_o}{2-y_o}\right)\right] - y_o \qquad (10)$$

for which the following trigonometric identity was invoked

$$\mathrm{TAN}\ (u-v) = \frac{\mathrm{TAN}\ (u) - \mathrm{TAN}\ (v)}{1 + [\mathrm{TAN}\ (u)]\cdot[\mathrm{TAN}\ (v)]}$$

The original problem concerned the quenching of 1000 gallons of an aqueous solution originally at 100°F. The physical properties of the solution were assumed to be the same as those for water. It was proposed to dump 1 ton of 1-inch ice cubes into the solution at quench time. The various problem parameters then assume the following values

$$W_t = (1000)\ (8.33) + 2000 = 10{,}330\ \text{lbs}$$

$$H_t = (1000)\ (8.33)\ (1.0)\ (100-32) - (144)\ (2000) = 278{,}000\ \text{BTU}$$

$$N = \frac{(2000)\ (12)^3}{(57.5)\ (1.0)^3} = 60{,}100$$

$$\alpha = \frac{(2)\ (200)}{(57.5)\ (1.0)} = 6.96\ \text{ft/hr}$$

$$\beta^3 = \frac{278{,}000}{(144)\ (57.5)\ (60{,}100)} = 0.558\cdot 10^{-3}\ \text{ft}^3$$

$$\beta = 0.0823\ \text{ft}$$

$$\gamma^3 = \frac{10{,}330}{(57.5)\ (60{,}100)} = 2.995\cdot 10^{-3}\ \text{ft}^3$$

$$\gamma = 0.1441\ \text{ft}$$

$$a = \frac{2.995\cdot 10^{-3}}{0.558\cdot 10^{-3}} = 5.37$$

$$c = \frac{6.96}{0.0823} = 84.6\ \text{hr}^{-1}$$

$$y_o = \frac{1}{(12)\ (0.0823)} = 1.012$$

Substitution of the last three quantities into Eq. (10) then results in the following time required for complete melting of the ice (solid phase)

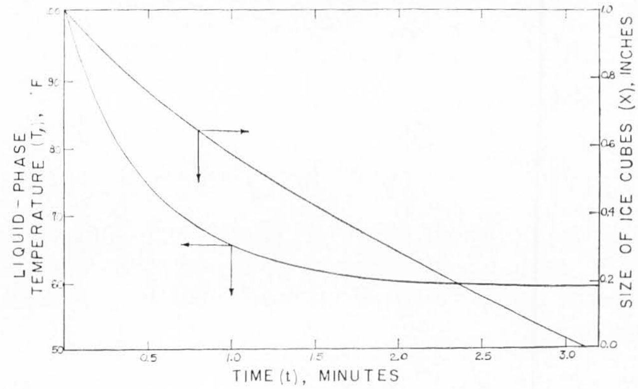


FIGURE 2. Time histories for the size of the ice cubes and the liquid-phase temperature.

$$t\ (x = 0) = 0.0514\ \text{hr} = 3.08\ \text{min.}$$

The temperature of the quenched aqueous phase at this time is given via algebraic solution of Eq. (5) with $W_s = 0$; the result is 58.9°F.

Complete time histories for the size of the ice cubes and the liquid-phase temperature for this particular system are presented in Fig. 2. □

NOMENCLATURE

- a — γ^3/β^3
- A — total surface area of unmelted solid phase, ft^2
- c — α/β
- c_p — heat capacity of liquid phase, BTU/lb–°F
- H — enthalpy, BTU
- N — total number of solid-phase cubes
- t — time, hr
- T — temperature, °F
- U — overall coefficient for heat transfer from the liquid phase to the solid phase, BTU/hr–ft^2–°F
- W — mass of material, lbs
- x — size of a solid-phase cube, ft
- y — x/β
- α — $2U/\rho c_p$
- β — $\sqrt[3]{H_t/\lambda\rho N}$
- γ — $\sqrt[3]{W_t/\rho N}$
- λ — latent heat of fusion of solid phase, BTU/lb
- ρ — density of solid phase, lbs/ft^3

SUBSCRIPTS

- l — liquid phase
- o — initial condition (t = 0)
- s — solid phase
- t — total (sum of liquid + solid phases)

SUPERSCRIPT

- o — initial condition (t = 0)