ChE classroom

LEVELS OF SIMPLIFICATION

THE USE OF ASSUMPTIONS, RESTRICTIONS, AND CONSTRAINTS IN ENGINEERING ANALYSIS

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The Navier-Stokes equations

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{v} \quad (1)$$

are exceedingly difficult to solve in their general form. Thus there is great motivation to search for plausible simplifications. One of these simplifications takes the form: **convective inertial effects are negligible.** This allows us to extract the linear version of Eq. (1) which is given by

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{v} \quad (2)$$

One could express this simplification as an equation, and there is some advantage in identifying it as a Level I *assumption* and expressing the idea as

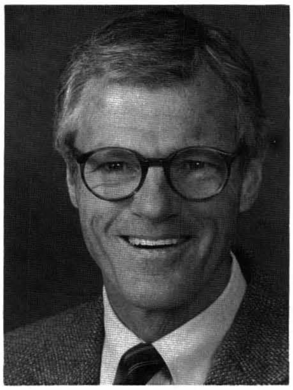
$$\text{Level I:} \qquad \rho\mathbf{v}\cdot\nabla\mathbf{v} = 0 \quad (3)$$

This type of statement indicates precisely what is being done in a mathematical sense, but it provides no basis for the action. For an engineer, it is more attractive to make a statement of the type: **convective inertial effects are small compared to viscous effects.** This leads to a Level II *restriction* of the form

$$\text{Level II:} \qquad \rho\mathbf{v}\cdot\nabla\mathbf{v} << \mu\nabla^2\mathbf{v} \quad (4)$$

In writing inequalities of this type it is understood that the comparison is being made between the absolute values of the terms under consideration.

Equation (4) has very definite advantages over Eq. (3) since a comment concerning the physics of the process under consideration has been made. While Eq. (4) tells the reader what must occur in order that Eq. (2) be valid, it does not indicate, in terms of the parameters of the problem, when it will occur. In order to determine this, one must be able to estimate the magnitude of the terms in Eq. (4). The treatment here will be brief since the details are given elsewhere [1]. We begin by expressing the velocity in terms of its magnitude and a unit tangent vector



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$$\mathbf{v} = v\boldsymbol{\lambda} \quad (5)$$

so that the convective inertial terms take the form

$$\mathbf{v}\cdot\nabla\mathbf{v} = v\boldsymbol{\lambda}\cdot\nabla\mathbf{v} \quad (6)$$

Since λ is a unit tangent vector to a streamline, we have

$$\mathbf{v}\cdot\nabla\mathbf{v} = v\,\frac{d\mathbf{v}}{ds} \quad (7)$$

where s is the arclength measured along a streamline. The derivative in Eq. (7) is estimated as [2, Sec. 2.9]

$$\frac{d\mathbf{v}}{ds} = \frac{\Delta\mathbf{v}}{L_\rho} \quad (8)$$

in which $\Delta\mathbf{v}$ represents the change in $\mathbf{v}$ that occurs along a streamline over the inertial length L_ρ. Use of Eq. (8) in Eq. (7) leads to an estimate of the convective inertial terms given by

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} = O\left(\frac{\rho v \Delta v}{L_\rho}\right) \tag{9}$$

It should be clear that a successful use of this estimate requires a reasonably good knowledge of the flow field. The viscous terms in Eq. (1) can be expressed as

$$\nabla^2 \mathbf{v} = \frac{\partial^2 \mathbf{v}}{\partial x^2} + \frac{\partial^2 \mathbf{v}}{\partial y^2} + \frac{\partial^2 \mathbf{v}}{\partial z^2} \tag{10}$$

and the associated order of magnitude is given by

$$\nabla^2 \mathbf{v} = O\left(\frac{\Delta \mathbf{v}|_x}{L_x^2}, \frac{\Delta \mathbf{v}|_y}{L_y^2}, \frac{\Delta \mathbf{v}|_z}{L_z^2}\right) \tag{11}$$

Here $\Delta\mathbf{v}|_x$ represents the change of $\mathbf{v}$ that occurs in the x-direction over the distance L_x, and the meaning of $\Delta\mathbf{v}|_y$ and $\Delta\mathbf{v}|_z$ is analogous for the y and z-directions. We represent the largest of the three terms on the right hand side of Eq. (11) as $\Delta\mathbf{v}/L_\mu^2$ and our estimate of the viscous terms takes the form

$$\nabla^2 \mathbf{v} = O\left(\frac{\Delta \mathbf{v}}{L_\mu^2}\right) \tag{12}$$

Here L_μ is referred to as the viscous length. For many cases the value of $\Delta\mathbf{v}$ in Eq. (12) is comparable to the value in Eq. (9) and this allows us to substitute Eqs. (9) and (12) into the inequality given by Eq. (4) in order to obtain

$$\frac{\rho v L_\mu^2}{\mu L_\rho} << 1 \tag{13}$$

Traditionally the Reynolds number is defined in terms of a length that is comparable to L_μ. Thus we use

$$Re = \frac{\rho v L_\mu}{\mu} \tag{14}$$

so that Eq. (13) takes the form

$$\text{Level III:} \quad Re \left(\frac{L_\mu}{L_\rho}\right) << 1 \tag{15}$$

Obviously this Level III constraint has a great deal more utility than the Level I assumption given by Eq. (3) for it allows one to decide *a priori* whether the analysis is applicable to a particular problem. When simplifying the Navier-Stokes equations on the basis of Eq. (15), one must remember Birkhoff's warning concerning the plausible intuitive hypothesis that "small causes produce small effects" [3].

While the route to Eq. (15) is straightforward, it is important to keep in mind that it is a *scalar* constraint associated with the magnitude of *vectors* and it must be used with care. In addition, it is crucial to understand that Eq. (15) has nothing to do with dimensional analysis, but is based entirely on the process of estimating the derivatives of the velocity that appear in Eq. (4). When the flow is turbulent, Eq. (1) must be time-averaged and Eq. (15) then applies to the time-averaged inertial and viscous terms. Estimating the Reynolds stress, $\overline{\rho \mathbf{v}' \cdot \nabla \mathbf{v}'}$, is more difficult than estimating $\rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}$ since a knowledge of the magnitude of $\mathbf{v}'$ and the turbulent length scale is required.

My thoughts concerning the various levels of simplification began to develop several years ago, and while the origin remains diffuse, I might place it in the early stages of an undergraduate heat transfer course.

Often it is difficult to develop Level III constraints for complex problems (think about the "perfectly mixed" stirred tank reactor), and one must settle for the type of simplification indicated by Eq. (3) in order to meet deadlines and complete required course material. From my point of view, the clearly stated Level I assumption is an acceptable simplification for it tells you what is being done and it reminds you that Level II restrictions and Level III constraints are waiting to be found. In addition, it should remind you that the analysis has an unspecified range of validity and that experiments and further analysis are in order.

SCENE

My thoughts concerning the various levels of simplification began to develop several years ago, and while the origin remains diffuse, I might place it in the early stages of an undergraduate heat transfer course. Because the subject under consideration was heat conduction, I began a lecture with $\mathbf{v} = 0$ and quickly discarded radiant energy transport to arrive at

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} \tag{16}$$

Since the assigned chapter and homework problems dealt with steady, one-dimensional heat conduction, we quickly moved to the following boundary value problem:

$$0 = \frac{d}{dx}\left(k \frac{dT}{dx}\right) \tag{17}$$

$$\text{B.C.1:} \quad T = T_0, \quad x = 0 \tag{18}$$

$$\text{B.C.2:} \quad T = T_1, \quad x = L \tag{19}$$

With the comment that "we can treat the thermal conductivity as constant," I was on the verge of presenting the classical result given by

$$T = T_0 + \left(T_1 - T_0\right)\left(\frac{x}{L}\right) \tag{20}$$

However, there was a flaw in my development. The title of the chapter under consideration indicated that we were to study the subject of steady, one-dimensional heat conduction, but it said nothing about the thermal conductivity being constant. One of the sages from the back row spotted the opening, and the traditional train of events was disrupted by the observation that "nothing is truly constant." Delighted to find that a portion of the back row was awake, I pursued Eq. (17) a bit further to arrive at

$$0 = \left(\frac{d^2T}{dx^2}\right) + \frac{1}{k}\left(\frac{\partial k}{\partial T}\right)\left(\frac{dT}{dx}\right)^2 \tag{21}$$

Since nonlinearities can be eliminated with impunity in an undergraduate class, I was willing to asume that $\partial k/\partial T$ was zero and move on to the desired result given by Eq. (20). However, the back row was warming to the task, and one of its occupants persisted with, "But nothing is really zero is it?" A reviewer of this article suggested that I should have counter-attacked with the Kirchhoff transformation [4, Sec. 2.16] so that Eqs. (17) through (19) could be expressed as

$$0 = \frac{d^2U}{dx^2} \tag{22}$$

$$\text{B.C. 1:} \qquad U = 0 \,, \quad x = 0 \tag{23}$$

$$\text{B.C. 2:} \qquad U = U_1 \,, \quad x = L \tag{24}$$

Here the transformed temperature is given by

$$U = \int_{\eta=T_0}^{\eta=T} \frac{k(\eta)}{k_0}\, d\eta \tag{25}$$

in which k_0 is the thermal conductivity at the temperature T_0. This approach would have avoided making the assumption that k was constant, but it would have delayed our arrival at Eq. (20) and the physical insight that can be gained from that result. While Eqs. (22) through (25) can provide an "exact" solution to the problem posed by Eqs. (17) through (19), we usually seek "approximate" solutions and often the approximations that we make are forced on our students by the title of the chapter and the name of the textbook.

In engineering education there is a conspiracy among students and faculty to base their assumptions on the title of the chapter currently under consideration. It is a game that is played with well established rules and most often both parties are loath to depart from the tradition. Students who are tempted to question the existence of the perfectly mixed stirred tank reactor are afraid that the instructor might plunge into a discourse on viscous dissipation, the Kolmogoroff length scale, and Damköhler numbers.* The terms of the treaty between students and faculty have been hammered out over the years, and by and large they work reasonably well. For example, can you imagine the difficulties of a study of fluid statics if it were preceded by the Level III constraints associated with

$$\rho \frac{\partial \mathbf{v}}{\partial t} << \rho \mathbf{g} \,, \quad \rho \mathbf{v}\cdot\nabla\mathbf{v} << \rho\mathbf{g} \,, \quad \mu\nabla^2\mathbf{v} << \rho\mathbf{g} \tag{26}$$

It is better to have a chapter entitled "Fluid Statics" so that the deck is cleared for an exploration of the pressure fields and forces associated with

$$0 = -\nabla p + \rho\mathbf{g} \tag{27}$$

Still, the question was posed from the back row, and it deserved an answer. Furthermore, it seemed to me that Eqs. (22) through (25) were most certainly *not* the answer, for the question was, in reality: How can you justify the simplification of Eq. (21) to arrive at

$$0 = \frac{d^2T}{dx^2} \tag{28}$$

Clearly the second term in Eq. (21) *cannot* be neglected on the basis of

$$\frac{1}{k}\left(\frac{\partial k}{\partial T}\right)\left(\frac{dT}{dx}\right)^2 << \frac{d^2T}{dx^2} \tag{29}$$

but surely conditions must exist for which the variation of the thermal conductivity is "small enough" so that Eq. (21) could be replaced by Eq. (28). This raises the question of "small relative to what?" and the following problem was devised to explore this question and to help students understand what is meant by *quasi-steady*.

SAMPLE PROBLEM

We consider the boundary value problem given by

$$\rho c_p \left(\frac{\partial T}{\partial t}\right) = k\left(\frac{\partial^2 T}{\partial x^2}\right) + \left(\frac{\partial k}{\partial T}\right)\left(\frac{\partial T}{\partial x}\right)^2 \tag{30}$$

$$\text{I.C} \qquad T = T_1 \,, \quad t = 0 \tag{31}$$

$$\text{B.C. 1:} \qquad T = T_1 + (T_0 - T_1)g(t) \,, \quad x = 0 \tag{32}$$

$$\text{B.C. 2:} \qquad T = T_1 \,, \quad x = L \tag{33}$$

*It is bad enough that the material would not be available in the text, but what is worse is that it would not be covered on the final!

Here g(t) is a function such that

$$g(t) = 0 , \quad t = 0 \tag{34a}$$

$$g(t) = 1 , \quad t = t^* \tag{34b}$$

The classic test piece in a study of separation of variables is associated with $t^* \to 0+$ and $(\partial k/\partial T) \to 0$, but in this case we should simply think of t^* as some characteristic time associated with the boundary condition at $x = 0$. Everyone knows that if t^* is "large enough" and if the variation of k with T is "small enough," the solution to this boundary value problem will yield the linear temperature profiles associated with Eq. (28). The Level I assumptions related to these conditions are

$$\text{Level Ia:} \qquad \left(\frac{\partial T}{\partial t}\right) = 0 \tag{35a}$$

$$\text{Level Ib:} \qquad \left(\frac{\partial k}{\partial T}\right) = 0 \tag{35b}$$

and one should be careful to identify the first of these as the *quasi-steady assumption.*

Our objective at this point is to develop the Level II and Level III restrictions and constraints that are associated with Eqs. (35a,b). Thus we seek to determine what is "large enough" and what is "small enough." If you have an idea that a satisfactory solution to Eqs. (30) through (34) might be given by*

$$T = T_1 + (T_0 - T_1)g(t)\left[1 - \left(\frac{x}{L}\right)\right] \tag{36}$$

the possibility can be explored by decomposing the temperature into the result represented by Eq. (36) and whatever else is left. One method of doing this is to arrange Eq. (30) as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\left(\frac{\partial T}{\partial t}\right) - \frac{1}{k}\left(\frac{\partial k}{\partial T}\right)\left(\frac{\partial T}{\partial x}\right)^2 \tag{37}$$

and to form the indefinite integral in order to obtain

$$\frac{\partial T}{\partial x} = \left.\frac{\partial T}{\partial x}\right|_{x=0} + \int_{\xi=0}^{\xi=x}\left[\frac{1}{\alpha}\left(\frac{\partial T}{\partial t}\right) - \frac{1}{k}\left(\frac{\partial k}{\partial T}\right)\left(\frac{\partial T}{\partial \xi}\right)^2\right]d\xi \tag{38}$$

Use of the definition

$$\Omega = \frac{1}{\alpha}\left(\frac{\partial T}{\partial t}\right) - \frac{1}{k}\left(\frac{\partial k}{\partial T}\right)\left(\frac{\partial T}{\partial \xi}\right)^2 \tag{39}$$

*This solution is obtained by using Eqs. (35a,b) to reduce Eq. (30) to the form given by Eq. (28). When Eq. (28) is solved subject to Eqs. (32) and (33), the solution given by Eq. (36) results.

along with a second integration leads to

$$T(x,t) = T\Big|_{x=0} + x\left(\frac{\partial T}{\partial x}\right)_{x=0} + \int_{\eta=0}^{\eta=x}\int_{\xi=0}^{\xi=\eta}\Omega\, d\xi\, d\eta \tag{40}$$

The boundary conditions given by Eqs. (32) and (33) can be used to evaluate the two constants in Eq. (40) and the general solution is given by

$$T(x,t) = T_1 + (T_0 - T_1)g(t)\left[1 - \left(\frac{x}{L}\right)\right] + \int_{\eta=0}^{\eta=x}\int_{\xi=0}^{\xi=\eta}\Omega\, d\xi\, d\eta - \left(\frac{x}{L}\right)\int_{\eta=0}^{\eta=L}\int_{\xi=0}^{\xi=\eta}\Omega\, d\xi\, d\eta \tag{41}$$

One should keep in mind that this is an exact representation for the temperature, but it is only useful when the integrals are negligible. The integrals in Eq. (41) can be estimated as

$$\int_{\eta=0}^{\eta=x}\int_{\xi=0}^{\xi=\eta}\Omega\, d\xi\, d\eta = \mathbf{O}(\Omega)\,\frac{x^2}{2} \tag{42a}$$

$$\int_{\eta=0}^{\eta=L}\int_{\xi=0}^{\xi=\eta}\Omega\, d\xi\, d\eta = \mathbf{O}(\Omega)\,\frac{L^2}{2} \tag{42b}$$

and use of these estimates in Eq. (41) leads to

$$T(x,t) = T_1 + (T_0 - T_1)g(t)\left[1 - \left(\frac{x}{L}\right)\right] + \frac{xL}{2}\left[1 - \left(\frac{x}{L}\right)\right]\mathbf{O}(\Omega) \tag{43}$$

We are now in a position to state that the solution for T(x,t) is qiven by

$$T(x,t) = T_1 + (T_0 - T_1)g(t)\left[1 - \left(\frac{x}{L}\right)\right] \tag{44}$$

provided that the following inequality is satisfied

$$(T_0 - T_1)g(t) >> \frac{L^2}{2}\,\mathbf{O}(\Omega) \tag{45}$$

This result allows us to replace the Level I assumptions given by Eqs. (35a,b) with the following Level II restrictions

$$\text{Level IIa:}\quad (T_0 - T_1)g(t) >> \mathbf{O}\left\{\frac{L^2}{\alpha}\left(\frac{\partial T}{\partial t}\right)\right\} \tag{46a}$$

$$\text{Level IIb:}\quad (T_0 - T_1)g(t) >> \mathbf{O}\left\{\frac{L^2}{k}\left(\frac{\partial k}{\partial T}\right)\left(\frac{\partial T}{\partial x}\right)^2\right\} \tag{46b}$$

Here we are beginning to see how "long" one must wait before the solution becomes quasi-steady, and how "small" the variation of the thermal conductivity must be in order that the last term in Eq. (30) can be discarded.

In order to proceed further, we must be willing to estimate the derivatives that appear in Eqs. (46a,b), and in this development we will be satisfied with the rather crude estimates given by [2, Sec. 2.9].

$$\frac{\partial T}{\partial t} = \mathbf{O}\left\{\frac{T|_{x=0} - T|_{x=L}}{t}\right\} = \mathbf{O}\left\{\frac{(T_0 - T_1)g(t)}{t}\right\}, \quad t \geq t^* \tag{47a}$$

$$\frac{\partial T}{\partial x} = \mathbf{O}\left\{\frac{T|_{x=0} - T|_{x=L}}{L}\right\} = \mathbf{O}\left\{\frac{(T_0 - T_1)g(t)}{L}\right\} \tag{47b}$$

This aspect of the problem could be considered more carefully by introducing the thermal boundary layer thickness; however, we are interested in knowing under what circumstances Eq. (44) is valid and the estimates given by Eqs. (47a,b) are consistent with that objective. When Eqs. (47a,b) are used in Eqs. (46a,b) we obtain the Level III constraints given by

$$\text{Level IIIa:}\quad \frac{\alpha t}{L^2} >> 1, \quad t \geq t^* \tag{48a}$$

$$\text{Level IIIb:}\quad \frac{1}{k}\left(\frac{\partial k}{\partial T}\right)(T_0 - T_1)g(t) << 1 \tag{48b}$$

The first of these clearly indicates that the process will be quasi-steady when t^* is large compared to L^2/α and an exact solution of the boundary value problem will indicate that this is a conservative constraint. Since g(t) has an upperbound of one, Eq. (48b) can be replaced by

$$\text{Level IIIb:}\quad \frac{1}{k}\left(\frac{\partial k}{\partial T}\right)(T_0 - T_1) << 1 \tag{48c}$$

While the results given by Eqs. (48a,b,c) are something that "everyone knows," not everyone knows how to arrive at these constraints without solving the full boundary value problem and exploring special cases. In addition, the identification of various levels of simplification is an important concept to bring to the attention of students, for it allows us to move quickly to certain simple engineering solutions while reminding us of our obligation to be more thorough when time permits or necessity demands. Following up on our obligations is sometimes easy to do. For example, in the typical heat transfer course transient processes are always studied, and when exact solutions are available it is easy to remind students of prior constraints that were developed on the basis of order of magnitude analysis. In the study of transient heat conduction in a flat plate one finds that Eq. (48a) can be replaced with $\alpha t/L^2 \geq 1$, thus providing a clear indication that the original estimation was overly severe. To support the result given by Eq. (48c), a homework problem associated with Eqs. (22) through (25) does rather nicely. The process of following order of magnitude estimates with exact solutions is an attractive method of encouraging students to develop their own assumptions, restrictions and constraints. As they gain confidence in this process, chapter titles become guidelines for the voyage rather than constraints for the next exam.

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