The object of this column is to enhance our readers' collection of interesting and novel problems in chemical engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class, or in a new light, or that can be assigned as a novel home problem, are requested, as well as those that are more traditional in nature and which elucidate difficult concepts. Please submit them to Professors James O. Wilkes and Mark A. Burns, Chemical Engineering Department, University of Michigan, Ann Arbor, MI 48109-2136.

MORE APPLIED MATH PROBLEMS ON VESSEL DRAINING

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Processing and storage vessels in the chemical and allied industries come in a large variety of shapes. There are almost as many reasons for this variability as there are shapes; these reasons can include convenience, insulation requirements, land and material costs, safety considerations, tradition, *etc.* Drainage of such vessels through an orifice-type hole at the bottom of the vessel represents a class of non-linear, ordinary, first-order differential equations, amenable to analytical solution. Thus, from an academic standpoint, this category of practical applications provides engineering educators with a wide variety of useful problems in the area of applied mathematics.

Solutions to these drainage problems have appeared for many of the geometrical configurations that typically occur in practice. These solutions normally appear in trade journals or similar outlets. For example, in one of the earlier such articles[1] on this subject, formulas were summarized to compute the time requirements for emptying vessels of four different shapes: vertical cylinder, cone, horizontal cylinder (with flat ends), and sphere. Later articles gave similar formulas for draining elliptical vessel heads at the bottom of vertical cylinders,[2] elliptical saturator troughs (horizontal elliptical cylinders with flat ends),[3] and horizontal cylinders with elliptical dished heads or end.[4]

One can conceive of a number of other geometrical shapes for vessels or tanks. Admittedly, they might not occur often in the real world, but such configurations may be of some use for academic purposes, *e.g.*, examination or homework problems. Thus (and also in the interest of completeness) this brief article presents tank-drainage formulas for five new configurations: parallelepiped (or box), vertical elliptical cylinder, regular tetrahedron, pyramid (inverted), and paraboloid.

GENERAL CONSIDERATIONS

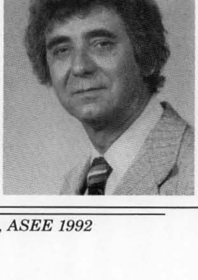
There are two fundamental engineering equations which must be invoked in the solution to any of these tank drainage problems. The first of these is a dynamic material balance for the liquid in the tank, which in this rather simple case merely reduces to the rate of accumulation being equal to the negative of the output rate

$$\frac{dV}{dt} = -q \tag{1}$$

or, more specifically

$$A\frac{dz}{dt} = -A_o v_2 \tag{2}$$

For the simpler geometric configurations (*e.g.*, vertical cylinders [circular or elliptical] and box), the



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cross-sectional area (A) of the liquid surface in the tank is a constant quantity and not a function of the variable liquid level (z). This results in a very tractable, non-linear differential equation.

The effluent liquid velocity (v_2) through the drain hole or orifice is determined from a mechanical energy balance, specifically between elevation points 1 and 2 in Figure 1. This results in the classical Bernoulli equation

$$\frac{P_1}{\rho} \cdot \frac{g_c}{g} + \frac{v_1^2}{2g} + Z_1 = \frac{P_2}{\rho} \cdot \frac{g_c}{g} + \frac{v_2^2}{2g} + Z_2 + h_\ell \quad (3)$$

Two assumptions are conventionally made at this point. The first of these assumes that the vessel is 1) vented to, and 2) drains to, the atmosphere, and thus $P_1 = P_2$. The second assumption asserts that the rate of change of the liquid level in the vessel (v_1) is negligible at all times in comparison with the liquid velocity through the drain hole (v_2). After replacing the variable elevation difference, $Z_1 - Z_2$, with the liquid level (z) in the tank, we have

$$v_2^2 = 2g(z - h_\ell) \quad (4)$$

When the orifice discharge equation is used, the fluid head loss (h_ℓ) due to friction is not explicitly calculated. Rather, an orifice discharge coefficient (C_o, generally less than unity) is introduced to attenuate the fluid head and compensate for this head loss

$$v_2 = C_o\sqrt{2gz} \quad (5)$$

As shown in most unit operations[5] and fluid flow textbooks, this quantity C_o is a function of the fluid velocity (as incorporated in the Reynolds number) and the downstream (orifice)/upstream diameter ratio, although a constant value is generally assumed for a given application. Typical values of C_o are between 0.60 and 1.0. Indeed, a value of 0.75 for this coefficient is reported in a recent article[6] describing an undergraduate experiment on efflux times through a drain hole at the bottom of a horizontal cylinder with flat ends. By way of information, civil engineers[7] know Eq. (5) (with C_o = 1) as the Torricelli theorem. Insertion of this latter expression into Eq. (2) then yields

$$A\frac{dz}{dt} = -C_o A_o\sqrt{2gz} \quad (6)$$

as the non-linear differential equation to be integrated.

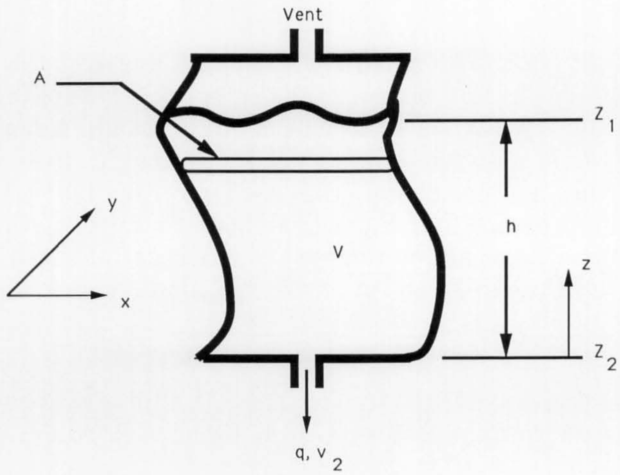

FIGURE 1. *Sketch of a vessel of arbitrary shape with a drain hole or orifice located at the bottom.*

Solutions to these drainage problems have appeared for many of the geometrical configurations that typically occur in practice. [They] normally appear in trade journals or similar outlets.

SPECIFIC CASES

The general integrated form of Eq. (6) can be written as

$$t = \frac{1}{C_o A_o\sqrt{2g}} \int_0^h \frac{A}{\sqrt{z}}\,dz \quad (7)$$

It should be emphasized at this point that the integral in the above expression cannot be replaced by the total volume (V) to be drained divided by the square root of the average fluid head over the total height (h) to be drained. That is

$$\int_0^h \frac{A}{\sqrt{z}}\,dz \neq \frac{\int_0^h A\,dz}{\frac{\int_0^h \sqrt{z}\,dz}{h}} = \frac{V}{\frac{2}{3}\sqrt{h}} \quad (8)$$

where the denominator represents the average fluid head. When A = const, the integration of Eq. (7) yields

$$t = \frac{2A\sqrt{h}}{C_o A_o\sqrt{2g}} \quad (9)$$

as the general equation for the efflux time to completely drain a tank whose cross-sectional area is not a function of height.

Parallelepipeds

This first simple case will employ Eq. (9). We consider a rectangular parallelepiped with a length and width (both in the horizontal plane) of a and b units, respectively. In this case, A = ab, and the efflux time from Eq. (9) is

$$t = \frac{2ab\sqrt{h}}{C_0 A_0 \sqrt{2g}} \tag{10}$$

Clearly, in the case of a square (a = b) parallelepiped, one merely replaces the product ab in Eq. (10) with a^2.

Vertical Elliptical Cylinder

This is another case wherein the surface area (A) formed by the liquid level is a constant, namely πab, where a and b are the lengths of the major and minor axes (again both in the horizontal plane), respectively, of the elliptical cross-section. Equation (9) in this case then becomes

$$t = \frac{2\pi ab\sqrt{h}}{C_0 A_0 \sqrt{2g}} \tag{11}$$

and in the special case of a vertical circular cylinder (a = b = D/2), Eq. (11) reduces to the equation presented earlier by Foster.[1]

Regular Tetrahedron

We consider here only the case of a regular tetrahedron with four equilateral triangular surfaces and with the drain hole located at a bottom vertex opposite the top triangle in the horizontal plane. The length of any edge of this figure is denoted by a. By application of the Pythagorean theorem, the height of any one of these triangles is equal to $a\sqrt{3}/2$. The total height of this figure is determined to be $a\sqrt{2/3}$ from a second application of this theorem, and then, from similar triangles, the cross-sectional area (A) of the liquid surface at any level z is given as $\left(3\sqrt{3}/8\right)z^2$. Lastly, insertion of this result into Eq. (7), followed by integration, then yields for the efflux time

$$t = \frac{3\sqrt{3}\,h^{5/2}}{20\,C_0 A_0 \sqrt{2g}} \tag{12}$$

Note the interesting result here that the edge length (a) does not appear in the above formula. It is clear, however, that the initial liquid level (h) cannot be selected in a manner inconsistent with a given such length; that is, h cannot exceed $a\sqrt{2/3}$. There is also the not-surprising result of h appearing to the power of 5/2 in the integrated expression of Eq. (12), which is consistent with Foster's result[1] for a conical tank.

Inverted Pyramid

This case of an inverted pyramid, like the tetrahedron above, is also similar to the case of an inverted cone. The drain hole is at the bottom vertex of the inverted pyramid, the total height of which is equal to c. A rectangular cross-section is assumed for generality. Thus, let a and b (both in the horizontal plane) represent the length and depth, respectively, of the pyramid at its top. Two successive applications of similar triangles yield $(ab/c^2)z^2$ as the expression for the area (A) of the liquid level at any elevation z. Integration of Eq. (7) with this expression for the area then gives

$$t = \frac{2ab\,h^{5/2}}{5c^2 C_0 A_0 \sqrt{2g}} \tag{13}$$

as the expression for the time for complete drainage. We note again the appearance of h to the power of 5/2 in Eq. (13), as in the preceding case. The product ab in this equation is merely replaced by a^2 in the case of a regular pyramid with a square side length of a.

Paraboloid

This last case examines an elliptical (again for generality) paraboloid of total height c. As with the vertical elliptical cylinder, a and b here represent the lengths of the major and minor axes, respectively, of the ellipse in the horizontal cross-section at the top of the paraboloid. The equation for this figure then becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \tag{14}$$

It should be noted that in many mathematical handbooks and textbooks the right-hand side of the above equation is written as cz or 2cz, in which case c would have the units of reciprocal length. In any event, any horizontal cross-section of this figure is elliptical, and from Eq. (14) it follows that the lengths of the major and minor axes of any such intermediate ellipse at an elevation of z are equal to

$$a\sqrt{z/c} \qquad \text{and} \qquad b\sqrt{z/c}$$

respectively. Thus, the area of this ellipse becomes $\pi abz/c$, and the resulting efflux time formula is

$$t = \frac{2\pi ab h^{3/2}}{3cC_0 A_0 \sqrt{2g}} \tag{15}$$

Finally, Eq. (15) becomes

$$t = \frac{\pi D^2 h^{3/2}}{6 c C_0 A_0 \sqrt{2g}} \tag{16}$$

as the expression for the efflux time in the special case of a circular (a = b = D/2) paraboloid.

NOMENCLATURE

- A cross-sectional area of the liquid level in a tank at any time, $\mathbf{L}^2$
- A_o cross-sectional area of the drain hole or orifice, $\mathbf{L}^2$
- a length of a rectangle, edge of a regular tetrahedron, or major axis of an ellipse, **L**
- b width of a rectangle or minor axis of an ellipse, **L**
- C_o orifice discharge coefficient
- c height of a pyramid or paraboloid, **L**
- D diameter of a circle, **L**
- **F** force unit
- g acceleration due to gravity, $\mathbf{L/T}^2$
- g_c conversion factor, $\mathbf{ML/FT}^2$
- h initial height of liquid in a tank, **L**
- h_ℓ fluid head loss due to friction, **L**
- **L** length unit
- **M** mass unit
- P pressure, $\mathbf{F/L}^2$
- q liquid volumetric flow rate out of a tank, $\mathbf{L}^3/\mathbf{T}$
- **T** time unit
- t time, **T**
- v liquid velocity, **L/T**
- x length coordinate in the horizontal plane, **L**
- y width or depth coordinate in the horizontal plane, **L**
- Z vertical elevation, **L**
- z variable elevation of the liquid level in a tank, **L**

Greek Letters

- π number pi (3.14159...)
- ρ liquid density, $\mathbf{M/L}^3$

Subscripts

- 1 liquid surface in the tank at any time
- 2 bottom of tank (at drain hole)

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ChE ***book review***

CHEMICAL KINETICS AND DYNAMICS

by Jeffrey I. Steinfeld, Joseph S. Francisco, and William L. Hase

Prentice Hall, New York; 548 pages, $48.75 (1989)

Reviewed by
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This book is a text intended for use in courses on chemical kinetics at the advanced undergraduate and graduate level. It covers a broad range of subjects in empirical (macroscopic) chemical kinetics, the kinetics of elementary reactions, the quantum state (microscopic) approach known as chemical dynamics, and the connections between them. Some background in thermodynamics, quantum and statistical mechanics, and kinetics at the level of an introductory course in physical chemistry is assumed, making it a suitable text for chemical engineering students.

The book consists of fifteen chapters and three appendices. After each chapter there is a list of references, a bibliography, and a number of problems. The references and bibliography will be useful for those seeking entry into the literature of various topics in chemical kinetics.

The book is unusually broad in its coverage, dealing with a number of subjects that are usually included only in more specialized texts, but which have now become commonplace in kinetics practice. Chapter 1 is conventional in its treatment of elementary concepts and definitions, but Chapter 2, dealing with complex reactions, goes beyond the usual presentation of analytical solutions to coupled sets of ordinary differential equations to discuss applications of Laplace transforms, matrix methods, numerical methods (Euler, Runge-Kutta, predictor-corrector) and stochastic methods. Computer programs for Runge-Kutta integration and Monte Carlo simulation are included. Chapter 3, on kinetic measurements, emphasizes modern instrumental methods for direct detection of reactive intermediates and the treatment of kinetic data. Sensitivity analysis, another subject not normally covered in introductory texts but which is of enormous help in understanding a mechanism, is introduced in this chapter.

Chapter 4 deals with reactions in solution and

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