The object of this column is to enhance our readers' collection of interesting and novel problems in chemical engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class, or in a new light, or that can be assigned as a novel home problem, are requested, as well as those that are more traditional in nature and which elucidate difficult concepts. Please submit them to Professors James O. Wilkes and Mark A. Burns, Chemical Engineering Department, University of Michigan, Ann Arbor, MI 48109-2136.

THREE PROBLEMS IN FLUID MECHANICS

JAMES O. WILKES, STACY G. BIKE
University of Michigan
Ann Arbor, MI 48109-2136

We present here (and solve) two homework problems that we have developed in the undergraduate chemical engineering fluid mechanics course at the University of Michigan. The first problem involves a fundamental principle of hydrostatics and requires thoughtful but simple reasoning for its solution, while the second problem is a good illustration of the application of potential-flow principles. A third problem is also presented, but is left for the reader to solve. The course is our second required undergraduate course, taken in the second term of the sophomore year, after thermodynamics (mass and energy balances). After trying a few textbooks, we (and our students) have opted instead for an extensive set of course notes that we have written and typeset. We always attempt to set problems that apply the principles of fluid mechanics to practical situations, albeit simplified in some cases.

The authors are both faculty members in the Department of Chemical Engineering at the University of Michigan. ***James O. Wilkes****, who is also Assistant Dean of the College of Engineering, has current research interests in the flow of paint films and injection-molding of polymer composites.* ***Stacey G. Bike*** *received her PhD from Carnegie Mellon University in 1988, and conducts research in the area of colloid science, including the fluid mechanics of colloidal dispersions and the rheological characterization of coatings.*

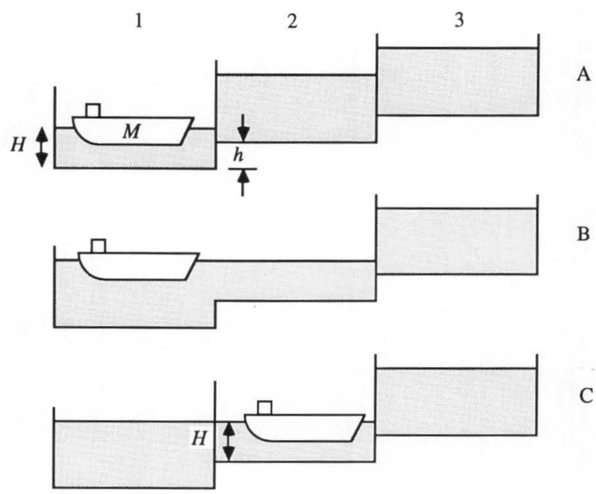

Figure 1. *Ship moving through locks.*

PROBLEM 1

Water Supply for a Ship Moving Through Locks

A ship of mass M travels uphill through a series of identical rectangular locks, each of equal superficial (birds-eye view) area, A, and elevation increase, h. The steps involved in moving from one lock to the next (1 to 2, for example) are shown as A-B-C in Figure 1. The lock at the top of the hill is supplied by a naturally occurring source of water of density ρ. Initially (A), the ship is isolated in lock 1, which has a depth of water H. The gate between locks 1 and 2 is then opened (B), equalizing the depths of water in the two locks. Finally (C), the ship moves into lock 2 and the gate is closed behind it.

1. Derive an expression for the increase in mass of water in lock 1 for the sequence shown, in terms

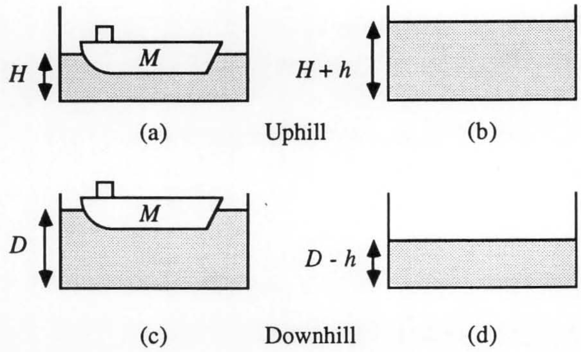

***Figure 2.** Ship moving from one lock to the next.*

of some or all of the variables M, H, h, A, ρ, and g.

2. If, after reaching the top of the hill, the ship descends through a similar series of locks to its original elevation, again derive an expression for the mass of water gained by a lock from the lock immediately above it. In this case, the initial depth in the uppermost lock will be D (greater than H).
3. Does the mass of water to be supplied depend on the mass of the ship if: (a) the ship travels only uphill, (b) the ship travels uphill, then downhill? Explain your answer.

SOLUTION

1. First, examine the ship as it travels uphill. As it passes from one lock to the next (say, from lock 1 to lock 2), the new depth of water in lock 2 must be H—exactly the same as it was in lock 1. The depth of the water remaining in lock 1 is therefore H + h. Figure 2 shows two appearances of lock 1: (a) first, when the ship is still in it, and (b) after the ship has moved into lock 2. Now examine the corresponding masses of water in lock 1 under these two conditions:
   - **(a)** From Archimedes' law, the weight of the water displaced by the floating ship is the weight of the ship itself, namely Mg. Therefore, when the ship is still in the lock, the *mass* of water displaced by the ship is M, so the mass of water in the lock is ρAH - M.
   - **(b)** After the ship has moved out of lock 1, the lock subsequently contains a mass of water ρA(H + h).

     Hence, the mass of water to be supplied is the difference between these two quantities:

$$\rho A(H+h)-(\rho AH-M)=\rho Ah+M \tag{1}$$

2. When the ship is proceeding downhill, as shown in Figures 2(c) and (d), the amount of water lost from the higher lock is likewise

$$(\rho AD-M)-\rho A(D-h)=\rho Ah-M \tag{2}$$

3. In conclusion, we observe from the above that
   - The amount of water to be supplied is ρAh ± M, depending on whether the ship is proceeding uphill or downhill, respectively.
   - Thus, the amount of water *does* depend on the mass of the ship, and is different for motion uphill or downhill.
   - If the ship navigates both uphill *and* downhill—as in traversing the Panama Canal, for example—the total water supply needed is 2ρAh, which is independent of the ship's mass. Thus, whether the *Queen Mary* or a rowboat is involved, the total water supply required is the same.

PROBLEM 2

Ground-Water Seepage

Figure 3 shows the seepage of water through the ground under a dam, caused by the excess pressure P (beyond that naturally occurring in the absence of the impounded water) that arises from the buildup of water behind the dam, which has (underground) a semi-circular base of radius r_D.

1. Verify the following relation, which has been proposed for the (excess) pressure in the ground:

$$p=P\left(1-\frac{\theta}{\pi}\right) \tag{3}$$

2. Determine the streamlines for the flow.
3. Between points A and B, a large amount of copper-impregnated soil has been detected, with the possibility that some of this toxic metal may leach out and have adverse effects downstream of the dam. To help assess the extent of this danger,

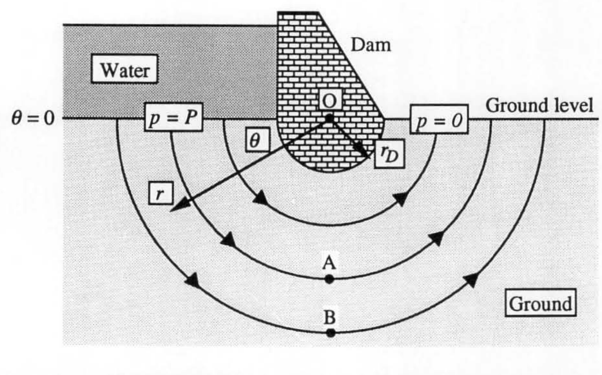

***Figure 3.** Seepage of water under a dam.*

derive an expression for the volumetric flow rate, Q, of water between A and B (per unit depth in the z-direction, normal to the plane of the diagram), in terms of P, K (the permeability of the ground), η (the viscosity of water), r_A, and r_B.

SOLUTION

1. Start by observing that the flow of water in the ground is governed by D'Arcy's law

$$\mathbf{v} = -\frac{K}{\eta}\nabla p \tag{4}$$

in which $\mathbf{v}$ is the (vector) superficial velocity and p is the pressure. By applying the continuity equation

$$\nabla \cdot \mathbf{v} = 0 \tag{5}$$

and assuming constant permeability K and viscosity η, we find that the pressure obeys Laplace's equation

$$\nabla^2 p = 0 \tag{6}$$

We are now reminded that the problem is essentially one of *potential flow*; indeed, the flow is *irrotational*, because the vorticity of a velocity that is proportional to the gradient of a scalar is zero, as may be checked by expanding $\nabla \times \mathbf{v} = \nabla \times \nabla p$ and discovering that it is a vector with three zero components.

Now examine the proposed pressure distribution by checking to see if it satisfies the following constraints:

(a) The conditions on pressure at the ground level. For $\theta = 0$ and π, Eq. (3) gives $p = P$ and $p = 0$, respectively, confirming the known pressures both upstream and downstream of the dam.

(b) Laplace's equation, $\nabla^2 p = 0$, in cylindrical ($r/\theta/z$) coordinates, in which all z derivatives are zero, is

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = 0 \tag{7}$$

The first term on the right-hand side of Eq. (7) is zero, because the proposed expression for p is independent of r. The second term is also zero, because p is only a linear function of θ. Thus, Laplace's equation is satisfied.

(c) Zero radial flow at the base of the dam. It will soon be seen that the radial velocity v_r is proportional to $\partial p/\partial r$, which is zero.

Thus, all constraints are satisfied by the proposed solution, which indicates that the pressure decreases linearly with the angle θ between the ground-level upstream and downstream values of P and zero, respectively.

2. The r and θ components of ∇p in cylindrical ($r/\theta/z$) coordinates are

$$(\nabla p)_r = \frac{\partial p}{\partial r} \qquad \text{and} \qquad (\nabla p)_\theta = \frac{1}{r}\frac{\partial p}{\partial \theta} \tag{8}$$

from which it follows (in conjunction with D'Arcy's law) that the radial and angular velocity components are

$$v_r = -\frac{K}{\eta}\frac{\partial p}{\partial r} = 0 \qquad \text{and} \qquad v_\theta = -\frac{K}{\eta}\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{KP}{\pi\eta r} \tag{9}$$

The corresponding expressions in terms of the stream function ψ are known to be

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad v_\theta = \frac{\partial \psi}{\partial r} \tag{10}$$

which, because of the minus sign in D'Arcy's law, are the negatives of those usually encountered. Substitution of the known values for v_r and v_θ from Eq. (9) into Eq. (10), and integrating, gives

$$\psi = f(r) \qquad \text{and} \qquad \psi = \frac{KP}{\pi\eta}\ell n\, r + g(\theta) \tag{11}$$

in which f(r) and g(θ) are *functions* of integration. The two expressions for the stream function in Eq. (11) must be compatible, so that $f(r) = (KP/\pi\eta)\, \ell n\, r$ and g(θ) is—at most—a constant, which may be taken as zero, giving

$$\psi = \frac{KP}{\pi\eta}\ell n\, r \tag{12}$$

Since the streamlines are contours of constant ψ, they must also be curves of constant r—that is, *semi-circles*, as shown in Figure 3. The isobars (or equipotentials) are, from Eq. (3), lines of constant θ and are orthogonal to the streamlines. If both streamlines and isobars were drawn, they would appear as the circular arcs and radial lines of a spider's web.

3. The flowrate between A and B (per unit depth, normal to the plane of the diagram) can be obtained in two ways. First, by definition of the stream function, it is simply the difference between ψ_A and ψ_B

$$Q = \psi_B - \psi_A = \frac{KP}{\pi\eta}\left(\ell n\, r_B - \ell n\, r_A\right) = \frac{KP}{\pi\eta}\ell n\frac{r_B}{r_A} \tag{13}$$

Second, the same result can be obtained by integrating the velocity between the two points:

$$Q = \int_A^B v_\theta\, dr = \int_A^B \frac{KP}{\pi\eta r}dr = \frac{KP}{\pi\eta}\ell n\frac{r_B}{r_A} \tag{14}$$

PROBLEM 3

Bubble Rise

We leave the reader with an intriguing problem that originated with our colleague, Professor (now *Emeritus*) M. Rasin Tek. As shown in Figure 4, a hollow vertical cylinder with *rigid* walls and of height H is closed at both ends and is filled with a volume, V, of an *incompressible* and *non-volatile* oil of density ρ at a uniform temperature T. A gauge registers the pressure at the top of the cylinder.

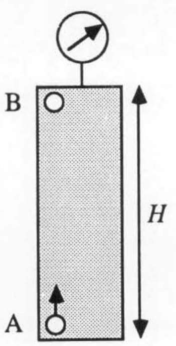

Figure 4. *Bubble rising in liquid in a closed cylinder.*

When a small spherical bubble of volume v initially adheres by surface tension to point A at the bottom of the cylinder, the *absolute* pressure at the top of the cylinder is p_0. The gas in the bubble is ideal, and has a molecular weight of M. The bubble is liberated by tapping on the cylinder and rises to point B at the top. Derive an expression in terms of any or all of the specified variables for the new absolute pressure p_1 at the top of the cylinder. Explain your answer carefully!

We leave the reader in suspense, requesting that he or she solve this problem. It is instructive for a fluid mechanics class because it shows that if you proceed methodically, the answer is deceptively simple. And, if you find it too easy, try it for the case when the oil is slightly *compressible,* with an isothermal compressibility β. ❐

ChE book review

CHEMICAL AND PROCESS THERMODYNAMICS
2nd Edition

by B.G. Kyle
Prentice Hall, Inc., Englewood Cliffs, NJ 07632

Reviewed by
E. Dendy Sloan
Colorado School of Mines

In this useful second edition, the author has avoided an encyclopedic "drink-of-water-from-a-fire-hydrant" approach to thermodynamics in favor of pedagogical digestion. The examples and problems with each chapter are well conceived, and a complete solutions manual is available. The text nomenclature and topic-ordering will seem familiar to professors teaching the topic.

Modern aspects of the book involve applications of classically stated fundamentals to environmental control, electrolytes, biochemicals, and electronic materials. Material on Jacobians, stability, and complex chemical equilibria go beyond topics found in many undergraduate texts.

A major asset of the book is its treatment of fluid properties. The author has eschewed the use of three-parameter corresponding states, providing graphic visualization of changes in compressibility and residual properties as a function of reduced temperature and pressure. An IBM-compatible floppy disk program (*ca.* 4000 lines) in the endpapers enables more accurate calculation of pure fluid properties (other than vapor pressure) through the Peng-Robinson equation of state (EOS).

A second major asset is the treatment of phase equilibria. After a brief treatment of principles, the author goes straight to applications, with advanced topics relegated to a later chapter. For example, in the first chapter on phase equilibria principles the author gives examples of activity coefficient hand calculations to optimize van Laar and Margules parameters, but a floppy disk program (*ca.* 5000 lines) is provided to either optimize or use Wilson equation parameters. Regular solution and UNIFAC treatments are delayed until the third chapter on phase equilibria.

The floppy disk programs represent one approach to introduce students to the foundations of ubiquitous flowsheeting programs in the profession. As such, a reader might wish for the unifying device of a Peng-Robinson EOS program applied to mixtures so that, for example, students could observe relative inaccuracies of an EOS versus activity coefficients for mixtures of alcohol+water or those of hydrocarbons.

Summing up, this book is a welcome addition for students learning undergraduate thermodynamics. If the author included an extension to molecular exposition and a final chapter on statistical thermodynamics, the book might also be a foundation to address the dearth of introductory graduate texts. ❐