ChE **classroom**

ChE APPLICATIONS OF ELLIPTIC INTEGRALS

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Elliptic functions and elliptic integrals remain a mystery to most chemical engineers—students, professors, and practitioners alike. Undoubtedly, this lack of familiarity derives from the classical absence of any significant general applications of these tools within the practice of chemical engineering. This situation is slowly changing, however, with recent developments in the area of fluid mechanics, particularly in relation to safety considerations. Thus, the purpose of this article is to present a brief exposition of the nature and genesis of elliptic functions and integrals, followed by a summary of some of their applications, with particular emphasis on chemical engineering problems.

ORIGIN OF ELLIPTIC FUNCTIONS

The fundamental elliptic functions actually derive from the analytical solution[1] to the parabolic partial differential equation describing unsteady-state heat conduction in one direction (z) through a flat plate π units thick. The initial condition on the temperature for this problem is assumed to be a Dirac function at the midplane of the plate ($z = \pi/2$). The boundary conditions for the spatial variable (at $z = 0$ and at $z = \pi$) may be either of two such conditions commonly invoked: 1) the two face temperatures are maintained at a constant value, or 2) the two faces of the plate are perfectly insulated, so that no heat transfer occurs at these two boundaries.

The analytical solutions to this problem may then be recast in terms of what are known as theta functions.[2] These latter are typically written as $\theta_i(z)$, where $i = 1, 2, 3, 4$ and $0 \le z \le \pi$. The three fundamental elliptic functions are then defined as various ratios of theta functions $[\theta_i(0), \theta_i(z)]$ and are denoted by sn(u), cn(u), and dn(u). The parameters z and u are related as follows: $z = u/[\theta_3(0)]^2$. A whole host of new elliptic functions then derive from these three fundamental elliptic functions, *e.g.*, ns(u), cs(u), nc(u), sc(u), dc(u), sd(u), etc., as well as a wide variety of mathematical expressions similar to trigonometric identities. Lastly, the various elliptic integrals are then defined in terms of these elliptic functions.

FUNDAMENTAL ELLIPTIC INTEGRALS

Perhaps a more straightforward manner in which to introduce the subject of elliptic integrals, however, is to describe one of the first problems that most likely led to their development. Thus, consider an ellipse, with its center at the origin of x-y coordinates (as in Figure 1), described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

where the lengths of its semi-major and semi-minor axes are given by a and b, respectively. What then is the value of its perimeter P (or periphery or circumference)? In the special case of a circle with $a = b = r$, the area (A) and circumference (C) are readily computed as πr^2 and $2\pi r$, respectively. Similarly, the area of an ellipse is readily determined from the calculus as πab, but the evaluation of its perimeter (P) is not so simple. Specifically, this latter quantity must be obtained by integration of the differential length of arc (ds) over the entire periphery of the ellipse.

For this purpose, it is convenient to convert x and y in Eq.

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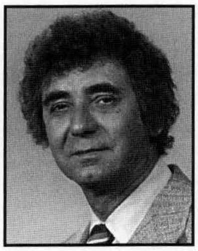

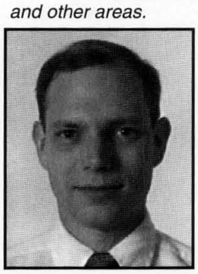

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(1) to parametric form, *e.g.,* to functions of the angular parameter θ:

$$x = a \sin\theta \qquad\qquad y = b\cos\theta \tag{2}$$

where, as also indicated by Figure 1, θ represents the eccentric angle measured from the minor axis b. We recall the definition of a differential length of arc as

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{3}$$

and let s here denote the arc length parameter measured clockwise along the curve from the end of the minor axis. Then, in terms of the angular parameter θ,

$$ds = \sqrt{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)}\, d\theta \tag{4}$$

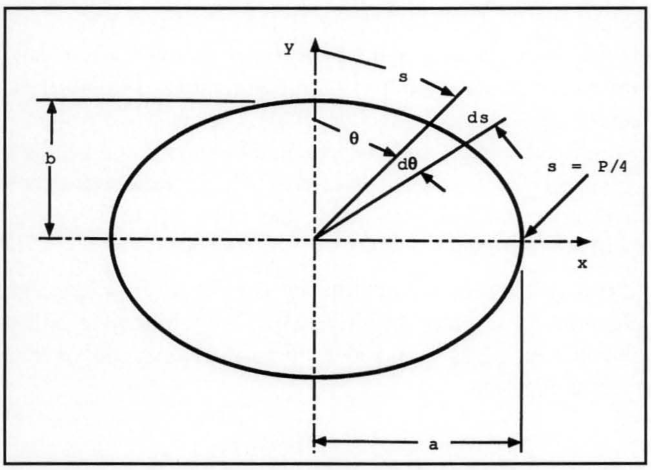

Figure 1. *Sketch of an ellipse for determination of the value of its perimeter P.*

Taking advantage of symmetry, it is clear that the total perimeter P of the ellipse is given as four times the perimeter of one quadrant, *e.g.,* from $\theta = 0$ to $\theta = \pi/2$. Thus, after replacing $\cos^2\theta$ with $\left(1-\sin^2\theta\right)$, we have

$$P = 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\, d\theta \tag{5}$$

as the expression for the perimeter of an ellipse. In Eq. (5),

$$e = \frac{\sqrt{a^2-b^2}}{a} \tag{6}$$

and is known as the eccentricity of the ellipse. More commonly, this quantity is referred to as the modulus k of the integral appearing in Eq. (5), which in turn is known as the complete (because of the fixed upper limit of $\pi/2$) elliptic integral of the second kind, generally denoted as $E(k)$. An incomplete elliptic integral of the second kind

$$E(k,\phi) = \int_0^{\phi}\sqrt{1-k^2\sin^2\theta}\, d\theta \tag{7}$$

has a second angular argument ϕ and obviously corresponds to incomplete integration ($\phi < \pi/2$) about the arc of the first quadrant in Figure 1.

The integral of Eq. (7) is one elliptic integral of three fundamental types. It can be shown[3] that any integral of the form

$$I = \int R\left(x, \sqrt{X}\right) dx \tag{8}$$

where X is a cubic or quartic in x and R denotes a rational function, can, by suitable linear transformations and reduction formulae, be expressed as the sum of a finite number of elementary integrals plus elliptic integrals of these three fundamental types. These types, in both incomplete and complete form are summarized in Table 1.

TABLE 1
Fundamental Elliptic Integrals (of the First, Second, and Third Kinds)

Kind	*Incomplete*	*Complete*
1.	$F(k,\phi) = \int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$	$K(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$
2.	$E(k,\phi) = \int_0^{\phi}\sqrt{1-k^2\sin^2\theta}\, d\theta$	$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\, d\theta$
3.	$\Pi(k,n,\phi) = \int_0^{\phi}\frac{d\theta}{\left(1+n\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}}$	$\Pi(k,n) = \int_0^{\pi/2}\frac{d\theta}{\left(1+n\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}}$

where
k = modulus of the elliptic integrals
ϕ = amplitude of the elliptic integrals
n = parameter in elliptic integrals of the third kind

There exist in the mathematical literature[4,5] extensive compilations of the transformations necessary to perform any integration involving the elliptic integrals associated with a given problem. Similarly, there are numerous handbooks[6-8] that tabulate numeric values of elliptic integrals to aid in the actual computations associated with such a problem.

PHYSICAL APPLICATIONS

Before proceeding on to technical applications of elliptic integrals closely associated with chemical engineering practice, we choose to summarize briefly some of the earliest physical problems whose solutions incorporate elliptic integrals. Most of these are of a mechanical nature.[9]

One of the early practical problems involving elliptic integrals pertains to determination of the oscillation period T of a pendulum of length L swinging through a circular arc. The solution of the ordinary differential equation describing this situation yields the expression[2,3,9]

$$T = \sqrt{\frac{L}{g}}\, K(k) \tag{9}$$

where g is the acceleration due to gravity. The modulus k of the elliptic integral in Eq. (9) is given by

$$k = \sqrt{\frac{h}{2L}} = \sin(\alpha/2) \tag{10}$$

Here, h represents the height of the maximum point to which the pendulum swings above its rest point, while α is the angular amplitude of the pendulum oscillations (corresponding to the height of this maximum point h).

Numerous other applications of elliptic integrals include characterization of planetary orbits under forces of attraction,[2] determination of the torque exerted by a mechanical brake,[9] and calculation of electrical current flow in a conducting plate.[2] And, of course, there is the natural geometric extension of computing the surface area of an ellipsoid. The general equation for the latter is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{11}$$

where $a > b > c$. It can be shown[2,3] that the surface area of such an ellipsoid in the general case is

$$S = 2\pi c^2 + \frac{2\pi ab}{\sin \upsilon}\left\{\left[\cos^2 \upsilon\right]\left[F(\upsilon, k)\right] + \left[\sin^2 \upsilon\right]\left[E(\upsilon, k)\right]\right\} \tag{12}$$

wherein the additional parameters υ and k are defined as

$$1 - c^2/a^2 = \sin^2 \upsilon \tag{13}$$

$$1 - c^2/b^2 = k^2 \sin^2 \upsilon \tag{14}$$

Simpler formulas (not requiring elliptic integrals) result in the special cases of 1) an oblate spheroid, for which $a = b$ (and hence $k = 1$), and 2) a prolate spheroid, for which $b = c$ (and hence $k = 0$). These various expressions for the surface areas of ellipsoids lead somewhat into the topic of applications of elliptic integrals in chemical engineering. Thus, from mass transfer studies,[10] for example, it is known that liquid droplets, such as are formed as the dispersed phase in liquid-liquid extraction, are often ellipsoidal in shape and their area is directly related to the rate of mass transfer.

CHEMICAL ENGINEERING APPLICATIONS

Most known applications of elliptic integrals in chemical engineering derive from fluid mechanics. A simple such application[9] which readily comes to mind is determination of the hydraulic radius (ratio of flow area to the wetted perimeter) for a pipe of elliptical shape, where a value for the perimeter of the elliptical cross-section is clearly required. Other early applications of elliptic integrals from fluid mechanics include derivation of the capillary curve for a fluid enclosed between two parallel vertical plates[9] and determination of the complex velocity potential for steady irrotational flow of liquid in two dimensions.[3]

Perhaps one of the more practical early uses of elliptic integrals is found in the case of liquid flow across weirs—traditionally more in the province of civil engineering but, with the recent advent of multifarious environmental concerns, often also employed by chemical engineers as measuring tools. Thus, classical civil engineering texts[11,12] present flow formulas for the more popular types of weirs, including rectangular and triangular (or V-notch weirs). While not employed extensively in this country (as they are in Europe), however, circular weirs for the measurement of liquid flow rates in open channels, such as ditches, flumes, and troughs, have the advantage that the crest can be turned and beveled with precision in a lathe. Moreover, this weir crest does not have to be leveled, and hence the point of zero flow is readily determined.

From the Bernoulli equation, the volumetric flow rate q as a function of the crest height h across a circular weir with a diameter of D, as depicted in Figure 2, is given by the integral equation

$$q = 2C_w\sqrt{2g}\int_0^h \sqrt{(D-z)z(h-z)}\,dz \tag{15}$$

where C_w is a weir discharge coefficient, accounting primarily for friction losses, much like an orifice discharge coefficient in closed channel flow measurement. In a 1957 paper, Stevens[13] found the analytical solution, incorporating elliptic integrals, for Eq. (15) to be of the form

$$q = \frac{4C_w\sqrt{2g}\,D^{5/2}}{15}\left[2\left(1-k^2+k^4\right)E(k) - \left(2-k^2\right)\left(1-k^2\right)K(k)\right] \tag{16}$$

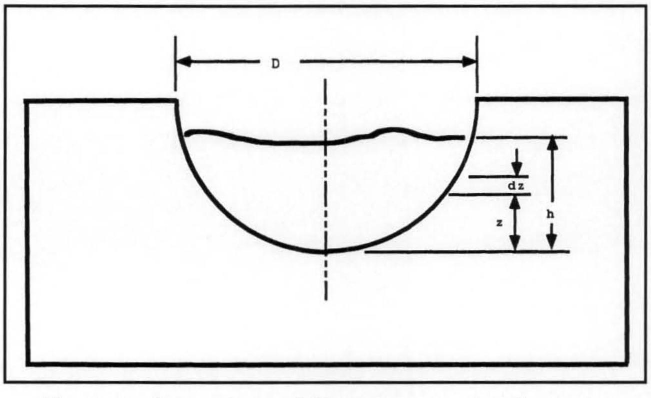

Figure 2. *Open channel flow across a circular weir.*

The modulus k of the elliptic integrals appearing in Eq. (16) is merely equal to $\sqrt{(h/D)}$. In his paper, Stevens also examined hundreds of experimental data points on water discharge rates from circular weirs. These data went back to the beginning of this century and were taken over the entire range of h/D from 0 to 1 on circular weirs up to three feet in diameter. An average value of the discharge coefficient C_w of 0.59 was determined from his analysis of these data.

Stevens' results were subsequently adapted to the problem of determining liquid overflow rates through circular openings in process and/or storage tanks.[14] Equation (16) thus applies equally to the problem of computing such discharge rates through circular apertures (or short discharge pipes), given the size of the opening and the liquid level therein. Indeed, Stevens[13] first became interested in this problem in conjunction with measuring the flow rate through a short pipe from a fishway into a power canal. In Reference [14], an approximate representation of Eq. (16), invoking the concept of relative volatility from vapor-liquid equilibria, was also developed and presented. Lastly, it comes as no surprise that this equation for the liquid flow rate across a circular weir is really just a special case for flow across an elliptical weir.[15]

The drainage of process vessels of many different shapes, such as cylindrical, spherical, and conical, represent conventional calculus problems, solutions to which have long been known.[16] To be sure, with the recently heightened interest in chemical process hazard analysis in addition to environmental issues, many of these drainage (or efflux) formulas have also appeared in recent textbooks on process safety.[17] It has been recently found that elliptic integrals (like Bessel functions in heat transfer) have a way of recurring in many fluid efflux problems with macroscopic circular geometries.

Thus, consider the problem of gravity drainage of a horizontal annulus, W units long, such as might be represented by the shell side of a double-pipe heat exchanger (see Figure 3). The inner and outer radii of this annulus are denoted by r_1 and r_2, respectively, while the drainage occurs through an aperture with a cross-sectional area of A_0 located along the bottom center line of the annulus. A constant value for the orifice discharge coefficient (*e.g.,* $C_0 = 0.61$) is assumed. Expressions for the drainage times required for the top and bottom thirds of this annulus (volumes I and III, respectively, in Figure 3) are readily obtained from earlier results for conventional horizontal circular cylinders.[16] But the drainage time requirement for the middle volume (t_{II}) of this annulus (that is, from the level of $h = r_2 + r_1$ down to $h = r_2 - r_1$) is given by an expression incorporating elliptic integrals[18]

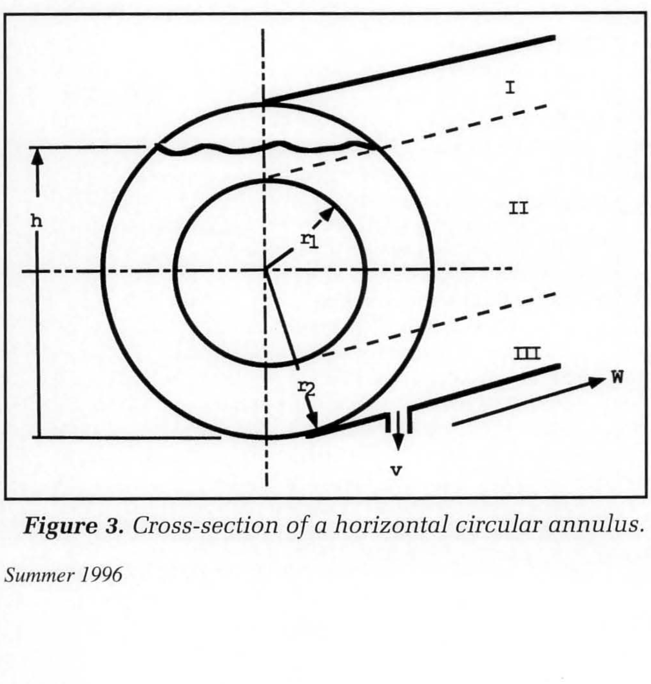

Figure 3. *Cross-section of a horizontal circular annulus.*

$$t_{II} = \frac{4\,W}{3C_0A_0\sqrt{2g}}\Big\{(r_2+r_1)^{3/2}-(r_2-r_1)^{3/2}$$

$$-2(r_2+r_1)^{1/2}\big[r_2E(k)-(r_2-r_1)K(k)\big]\Big\} \qquad (17)$$

where the modulus k in this case is given by

$$k = \sqrt{\frac{2\,r_1}{r_2+r_1}} \qquad (18)$$

The more general expression for partial drainage of this middle volume (II) of a horizontal annulus (*i.e.,* from some intermediate elevation and/or down to some other intermediate elevation, both within this middle volume) is considerably more complicated and specifically incorporates *incomplete* elliptic integrals of the first and second kinds.[18]

Until recently, most fluid efflux analyses pertained to intentional drainage from an opening at the bottom of a vessel. But now, because of increasing concerns about safety and loss prevention in the process industries, there exists a need for accurate formulas to compute fluid discharge and vessel emptying rates for an opening at an arbitrary elevation. Such a need may arise in analyzing an accident scenario resulting from a moving vehicle, *e.g.,* a forklift truck or an automated guided vehicle (AGV), being driven into the side of a vessel. Such analytical formulas were originally presented by Crowl[19] for spherical and vertical cylindrical vessels.

Subsequently, the following expression was developed[20] for the time t required for drainage of a horizontal cylindrical vessel, with a diameter of D and W units long, from an arbitrary initial liquid level of h_1 through a hole with a cross-sectional area of A_0 and located at an equally arbitrary elevation of h_0,

$$t = \frac{4\,W}{3C_0A_0\sqrt{2g}}\Big\{\sqrt{D}\big[(D-2\,h_0)E(\phi,k)+h_0F(\phi,k)\big]$$

$$+(2\,h_0+h_1-D)\sqrt{\frac{(D-h_1)(h_1-h_0)}{h_1}}\Big\} \qquad (19)$$

A sketch of this configuration is shown in Figure 4. The parameters of the incomplete elliptic integrals in Eq. (19) are

$$\phi = \sin^{-1}\sqrt{\frac{D(h_l - h_0)}{(D - h_0)h_l}} \quad (20)$$

and

$$k = \sqrt{\frac{D - h_0}{D}} \quad (21)$$

In this case, if the time required for the liquid level to fall from an initial elevation of h_l to some intermediate elevation h_i (or, equivalently, to discharge a given amount of material) is desired, two successive applications of Eqs. (19) and (20) can be employed for this purpose.

Recent interest has also arisen in the problem of exhausting process vessels through drain piping systems.[21] Thus, the case of pipeline drainage of horizontal cylindrical tanks also requires elliptic integrals.[22] Such a configuration is presented in Figure 5. In this instance, one is interested in the time required to drain the contents of a horizontal cylindrical vessel with a diameter of D and a length of W through a drain pipe system with an inside diameter of d, attached at the bottom center line of the vessel. This drain piping system originates at an elevation of h_0 units above the datum plane and has an equivalent length of L. Fully developed turbulent flow through this system is assumed, with a constant Moody friction factor of f.

With these assumptions, the resulting analytical solution[22] to this problem again incorporates (in the general case of incomplete drainage of a partially filled vessel) the incomplete elliptic integrals of the first $F(\phi, k)$ and second $E(\phi, k)$ kinds. The latter collapse down to their complete form for the special case of complete drainage of a completely filled horizontal circular cylinder through a drain piping system. Saturator troughs in the shape of horizontal semi-elliptical cylinders are employed extensively in the textile finishing industries. Not surprisingly, the solution to the problem of determining drainage times for such troughs through a piping system also invokes elliptic integrals.[23]

CONCLUSION

In this article, we have addressed the subject of elliptic integrals, including their origins and definitions. Early scientific applications of elliptic integrals, primarily from the physics area, were briefly summarized. Then, a number of such applications in chemical engineering, most of which are relatively recent in origin, were described (see

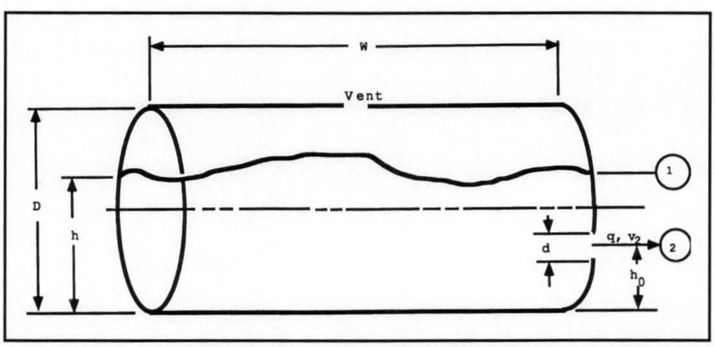

Figure 4. *Horizontal circular cylindrical vessel with a puncture hole in its side and resulting liquid drainage.*

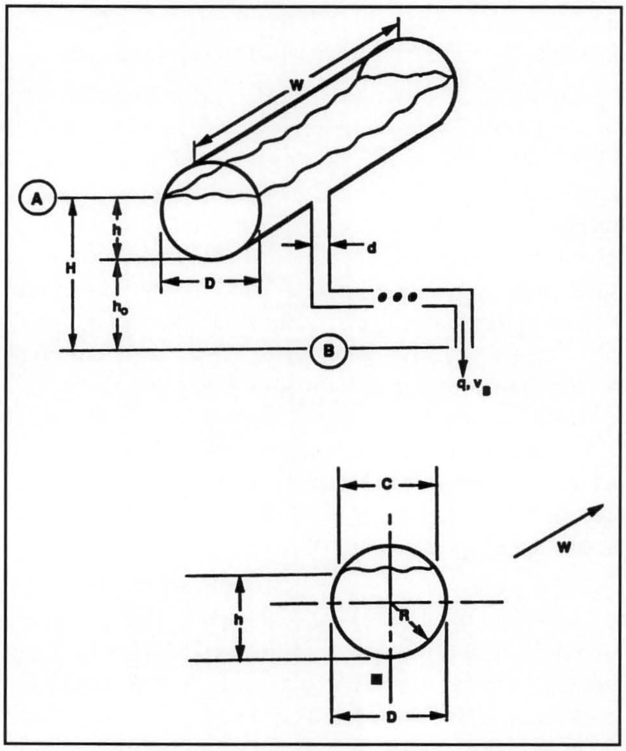

Figure 5. *Sketch of a horizontal circular cylindrical tank with drain piping.*

TABLE 2
Summary of Technical Problems with Elliptic Integral Solutions

Problem	*Reference(s)*
Physics Problems	
• Area of an ellipse	[2,9]
• Period of oscillation for a swinging pendulum	[2,3,9]
• Torque exerted by a mechanical brake	[9]
• Motion of a whirling chain or skipping rope	[2,3,9]
• Area of the surface of an ellipsoid	[2,3]
• Planetary orbits under laws of attraction	[2]
• Current flow in a rectangular conducting plate	[2]
• Electrostatics of a parallel plate capacitor	[2]
Chemical Engineering Problems	
• Hydraulic radius of an elliptical pipe	[9]
• Capillarity between two parallel vertical plates	[9]
• Steady irrotational liquid flow in two directions	[3]
• Fluid flow across circular weirs or openings	[13,14]
• Fluid flow across elliptical weirs or openings	[15]
• Bottom drainage of horizontal annuli	[18]
• Efflux from punctured horizontal cylinders	[20]
• Drainage of horizontal cylinders through piping	[22,23]

Table 2). Most of the chemical engineering applications of elliptic integrals to date have been in the fluid mechanics area.

NOMENCLATURE

A = surface area formed by the liquid level in a tank
A_0 = cross-sectional area of flow opening
a = length of semi-major axis of an ellipse
b = length of semi-minor axis of an ellipse
C = circumference of a circle; length of chord formed by a liquid level
C_0 = orifice discharge coefficient
C_w = weir discharge coefficient
c = length of third semi-axis of an ellipsoid
D = diameter of a circular tank or weir
d = diameter of a circle
d_0 = diameter of flow opening
E = elliptic integral (incomplete or complete) of the second kind
e = eccentricity of an ellipse $\left(=\left\{\left[a^2-b^2\right]^{1/2}\right\}/a\right)$
F = incomplete elliptic integral of the first kind
g = acceleration due to gravity
H = variable elevation of the liquid level in a tank above the outlet of drain piping
h = maximum elevation of a swinging pendulum above its rest point; variable elevation or height of the liquid level in a tank
h_1 = initial elevation or height of the liquid level in a tank
h_0 = elevation of a tank bottom above the outlet of drain piping
I = general integral of Eq. (8)
K = complete elliptic integral of the first kind
k = modulus of elliptic integrals; parameter in calculation of ellipsoidal surface areas, defined in Eq. (14)
L = equivalent length of piping
n = parameter of elliptic integrals of the third kind
P = perimeter of an ellipse
q = volumetric flow rate
R = rational function of x and $\sqrt{X}$ in Eq. (8); radius of a circular tank or weir
r = radius of a circle
S = surface area of an ellipsoid
s = length of arc
T = period of oscillation for a swinging pendulum
t = time
u = argument of elliptic functions
V = fluid volume
v = linear velocity
W = length of a horizontal cylinder
X = cubic or quartic function of x in Eq. (8)
x = arbitrary independent variable of integration; horizontal coordinate
y = vertical coordinate
z = thickness of a flat plate

Greek Letters

α = angular amplitude of oscillation of a pendulum
ϕ = amplitude of elliptic integrals
υ = parameter in calculation of ellipsoidal surface areas, defined in Eq. (13)
Π = elliptic integral (incomplete or complete) of the third kind
π = number pi (3.14159...)
θ = theta function; angular argument of elliptic integrals

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