DYNAMICS OF A STIRRED-TANK HEATER

Intuition and Analysis

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A common example in chemical engineering textbooks on dynamic modeling and control is the stirred-tank heater illustrated in Figure 1. If the tank is treated as perfectly mixed, it represents a simple process that allows for a straightforward derivation of the governing equations, and it helps to illustrate the use of constitutive equations in developing dynamic process models. The assumptions involved in the analysis are typically justified by common sense and practical insight. With them, one arrives at a model that can be used for demonstrating dynamic behavior through simulations and for testing feedback control concepts.

This problem has found its way into the majority of dynamics and control textbooks, and in these textbooks we observe three different approaches to the analysis. The first approach is represented by the assumption of constant flow rates (thus constant holdup), which simplifies the problem considerably.[1-7] Seborg, *et al.*,[8] derive the model equations but also focus on the constant holdup problem in the text. The second approach is by Harriot[9] and by Pollard,[10] who consider flow variations but still assume constant holdup. The book by Stephanopoulos[11] provides a third approach in which the holdup is not constant and perturbations are present in both inlet and outlet flowrates. This leads to the multivariable control of both the holdup and the temperature in the tank.

If the volume of fluid in the tank is allowed to vary (in response to variations in the *outlet* flowrate), the coupling between the mass and energy balances generates a curious, non-trivial problem that has not been fully recognized in the literature. Specifically, the dynamic behavior of the stirred tank heater appears to contradict, at least initially, our intuitive reasoning, leading to an incorrect interpretation of the expected behavior.

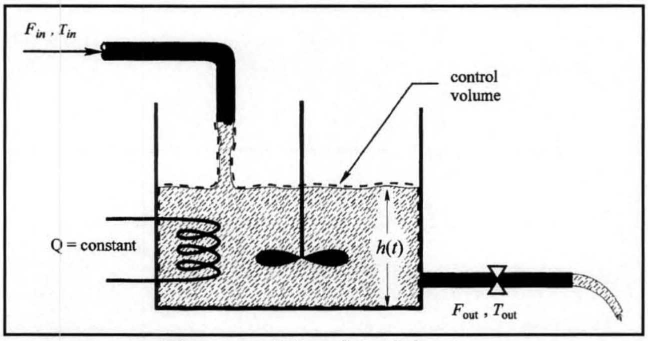

Figure 1. *Stirred-tank heater.*

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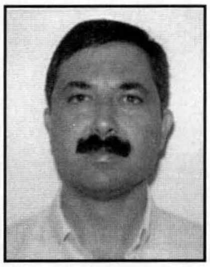

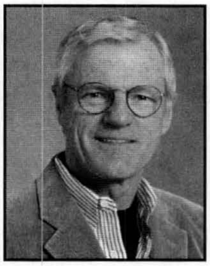

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In the next section, we first define the problem and offer an intuitive description of the dynamic behavior of the process. Next, in an effort to provide a rigorous explanation of why we should expect a dynamic response contrary to our intuition, we derive the dynamic model of the stirred-tank heater process, carefully delineating all the steps and the assumptions involved. Finally, we offer some insight into the correct intuitive interpretation of the analytical result.

PROBLEM DEFINITION

Figure 1 depicts a stirred-tank heat process. The purpose of this unit is to provide heating for a process stream, thereby increasing its temperature before being supplied to a downstream unit. The heat provided is denoted by Q. In the schematic diagram, F and T represent a stream flowrate and a stream temperature, respectively, and the subscripts "in" and "out" refer to the inlet and outlet streams. We assume that there is a valve placed at the outlet flow stream, which can be adjusted to affect the tank level.

► *Intuition*

Now, let us perform the following thought experiment. Assuming that the system is initially at steady state, we shall *increase the outlet stream flowrate,* thereby creating a transient process in which the level in the tank decreases, as illustrated in Figure 2. The heat supplied to the system, Q, remains constant, and we want to know how the temperature of the outlet stream responds to this change. Typically, intuition leads to the suggestion that the outlet stream temperature will *increase* since less mass is now being heated in the tank. If we *decrease the outlet stream flowrate,* the level in the tank will increase, as illustrated in Figure 3. The typical intuitive interpretation of this transient process is that the outlet stream temperature will *decrease* because more mass is now being heated in the tank.

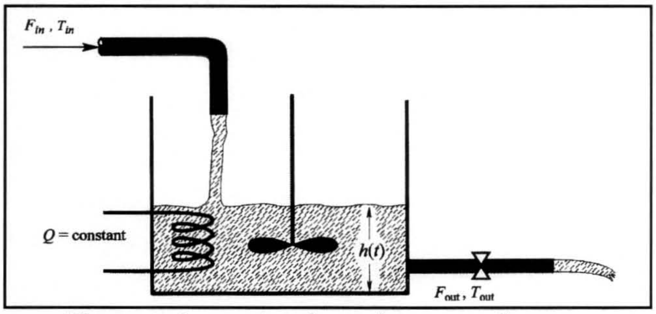

Figure 2. *Increasing the outlet stream flowrate.*

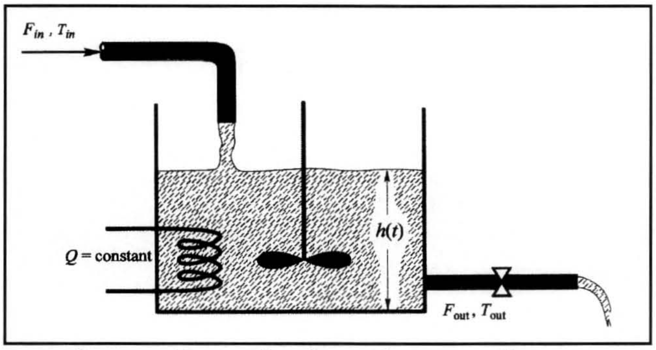

Figure 3. *Decreasing the outlet stream flowrate.*

Both these intuitive interpretations of the dynamic response of the stirred-tank heater are incorrect, as can be shown by a careful analysis of the system. We must note that if one changes the inlet flowrate instead of the outlet flowrate, the above reasoning leads to the correct answer, and thus there is a fundamental difference in the way each flowrate affects the temperature in the tank.

To summarize our survey of intuitive judgments concerning the influence of changing the outlet flowrate, we note that almost everyone believes that changing the outlet flowrate will change the outlet temperature. In addition, most believe that increasing the flowrate will cause an increase in the temperature, while decreasing the flowrate will cause a decrease in the temperature.

ANALYSIS

► *Modeling*

To develop the dynamic model of the stirred-tank heater, we make use of the macroscopic mass and thermal energy balances[12] for a moving control volume. These are given by

$$\frac{d}{dt}\int_{V(t)} \rho dV + \int_{A(t)} \rho(\mathbf{v}-\mathbf{w})\cdot\mathbf{n}dA = 0 \tag{1}$$

$$\frac{d}{dt}\int_{V(t)} \rho c_p\left(T-T_{ref}\right)dV + \int_{A(t)} \rho c_p\left(T-T_{ref}\right)(\mathbf{v}-\mathbf{w})\cdot\mathbf{n}dA = -\int_{A(t)}\left(\mathbf{q}+\mathbf{q}^R\right)\cdot\mathbf{n}dA + \int_{V(t)}\left(T\beta\frac{Dp}{Dt}+\boldsymbol{\tau}:\nabla\mathbf{v}\right)dV \tag{2}$$

in which the moving control volume, V(t), contains the fluid in the stirred tank as illustrated in Figure 1. In terms of the liquid depth, h(t), and the cross-sectional area of the tank, A, the control volume can be expressed as

$$V(t) = Ah(t) \tag{3}$$

At the gas-liquid interface, the kinematic condition requires that

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} \tag{4}$$

while at the liquid solid interface we have

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} = 0 \tag{5}$$

Use of these two conditions, and the assumption that the density can be treated as a constant, allows us to express the macroscopic mass balance as

$$A\frac{dh}{dt} = F_{in} - F_{out} \tag{6}$$

in which F_{in} and F_{out} represent the volumetric flow rates entering and leaving the system. To be explicit, we express F_{out} according to

$$F_{out} = \int_{A_{exit}} \mathbf{v}\cdot\mathbf{n}\,dA \tag{7}$$

In our treatment of the thermal energy balance, we neglect the reversible work, $T\beta\, Dp/Dt$, and the viscous dissipation, $\tau:\nabla\mathbf{v}$, so that Eq. 2 takes the form

$$\frac{d}{dt}\int_{V(t)} \rho c_p (T - T_{ref})\,dV + \int_{A(t)} \rho c_p (T - T_{ref})(\mathbf{v}-\mathbf{w})\cdot\mathbf{n}\,dA = -\int_{A(t)} \left(\mathbf{q}+\mathbf{q}^R\right)\cdot\mathbf{n}\,dA \qquad (8)$$

If the only significant heat transfer to the system is caused by the heater, the heat-transfer term on the right-hand side of this result can be expressed as

$$-\int_{A(t)} \left(\mathbf{q}+\mathbf{q}^R\right)\cdot\mathbf{n}\,dA = Q \qquad (9)$$

in which Q is positive when heat is transferred *to* the system. Assuming that the variations of ρ and c_p are negligible, and making use of Eqs. 3 through 5, allows us to write Eq. 8 in the form

$$\frac{d}{dt}\left[\rho c_p A h(t)\left(\langle T\rangle - T_{ref}\right)\right] = \rho c_p F_{in}\left(T_{in} - T_{ref}\right) - \rho c_p F_{out}\left(T_{out} - T_{ref}\right) + Q \qquad (10)$$

Here the volume averaged temperature, $\langle T\rangle$, is defined by

$$\langle T\rangle = \frac{1}{V(t)}\int_{V(t)} T\,dV \qquad (11)$$

and since ρc_p can be treated as a constant, Eq. 10 can be rearranged as

$$\frac{d}{dt}\left[A h(t)\left(\langle T\rangle - T_{ref}\right)\right] = F_{in}\left(T_{in} - T_{ref}\right) - F_{out}\left(T_{out} - T_{ref}\right) + \frac{Q}{\rho c_p} \qquad (12)$$

Carrying out the differentiation on the left-hand side leads to

$$A h(t)\frac{d\langle T\rangle}{dt} + \left(\langle T\rangle - T_{ref}\right) A\frac{dh}{dt} = F_{in}\left(T_{in} - T_{ref}\right) - F_{out}\left(T_{out} - T_{ref}\right) + \frac{Q}{\rho c_p} \qquad (13)$$

Equation 6 can be multiplied by $\langle T\rangle - T_{ref}$, leading to

$$A\frac{dh}{dt}\left(\langle T\rangle - T_{ref}\right) = F_{in}\left(\langle T\rangle - T_{ref}\right) - F_{out}\left(\langle T\rangle - T_{ref}\right) \qquad (14)$$

and when this result is subtracted from Eq. 13, we obtain a simplified form of the macroscopic thermal energy balance. The macroscopic mass and thermal energy balances represent the governing differential equations for the fluid depth and the volume-averaged temperature in the stirred tank. We list these two results and the initial conditions as

Mass

$$A\frac{dh}{dt} = F_{in} - F_{out} \qquad (15)$$

Thermal Energy

$$A h(t)\frac{d\langle T\rangle}{dt} = F_{in}\left(T_{in} - \langle T\rangle\right) + \frac{Q}{\rho c_p} \qquad (16)$$

Initial Conditions

$$h = h^\circ \qquad \langle T\rangle = \langle T\rangle^\circ \qquad F_{out} = F_{in} \qquad t = 0 \qquad (17)$$

The driving force for the dynamic behavior is the *outlet stream* flowrate, which is a function of time, *i.e.,*

$$F_{out} = F_{in} + \Delta F(t) \qquad t \ge 0 \qquad (18)$$

At this point, we identify the state variables and constant parameters as

State Variables: h and $\langle T\rangle$

Constant Parameters: A, ρc_p, Q, F_{in}, and T_{in}

It may not be obvious on the basis of Eqs. (15) through (18), but the solution for the volume-averaged temperature for this process is given by

$$\langle T\rangle = \langle T\rangle^\circ \qquad t \ge 0 \qquad (19)$$

► *Solution*

In practice, the change in the outlet stream flowrate, ΔF, is a function of h°-h(t), and thus the outlet stream flowrate can be expressed by

$$F_{out} = F_{in} + \Delta F\left[h^\circ - h(t)\right] \qquad t \ge 0 \qquad (20)$$

Use of this result in the macroscopic mass balance given by Eq. (15) leads to

$$A\frac{dh}{dt} = -\Delta F\left[h^\circ - h(t)\right] \qquad (21)$$

and this can be solved subject to the initial condition

$$\text{I.C.} \qquad h = h^\circ \qquad t = 0 \qquad (22)$$

in order to determine h(t). The solution for the fluid depth as a function of time can then be used with Eq. (16) to determine the volume-averaged temperature in the tank. The initial steady-state condition of the stirred tank must satisfy the following form of the macroscopic thermal energy balance

$$0 = F_{in}\left(T_{in} - \langle T\rangle^\circ\right) + \frac{Q}{\rho c_p} \qquad (23)$$

This result can be subtracted from Eq. (16) to obtain

$$A h(t)\frac{d\langle T\rangle}{dt} = -F_{in}\left(\langle T\rangle - \langle T\rangle^\circ\right) \qquad (24)$$

If we identify the temperature difference according to

$$\Theta = \langle T\rangle - \langle T\rangle^\circ \qquad (25)$$

the initial value problem for Θ takes the form

$$f(t)\frac{d\Theta}{dt} = -\Theta \qquad (26a)$$

$$\text{I.C.} \qquad \Theta = 0 \qquad t = 0 \qquad (26b)$$

Here the time dependent function, f(t), is given by

$$f(t) = \frac{A h(t)}{F_{in}} \qquad (27)$$

and the solution to Eqs. (26a) and (26b) is

$$\Theta = 0 \qquad t \ge 0 \qquad (28)$$

This leads to the result listed earlier as Eq. (19), which is often considered to be counter-intuitive.

The forms of Eqs. (26a,b) and the solution given by Eq. (28) usually create a little skepticism; however, we can transform the initial value problem to a more familiar form in order to make the solution more appealing. We begin by letting τ be a function of time defined by

$$\tau = \int_{\eta=0}^{\eta=t} f(\eta)^{-1} d\eta \qquad f(t) \neq 0 \tag{29}$$

This transformation leads to

$$\frac{d\Theta}{dt} = \frac{d\Theta}{d\tau}\frac{d\tau}{dt} = \frac{d\Theta}{d\tau} f(t)^{-1} \tag{30}$$

and the initial-value problem given by Eqs. (26a,b) takes the form

$$\frac{d\Theta}{d\tau} = -\Theta \tag{31a}$$

$$\text{I.C.} \qquad \Theta = 0 \qquad \tau = 0 \tag{31b}$$

Clearly, the solution to this initial-value problem is given by Eq. (28).

DISCUSSION OF DYNAMIC BEHAVIOR

The source of the seemingly counter-intuitive behavior of the tank temperature when the outlet stream flowrate is changed lies in the key assumption of *complete mixing* in the tank. To provide a purely intuitive confirmation of either Eq. (19) or Eq. (28), we construct special processes in which the fluid depth in the tank either decreases or increases.

Case of Outlet Stream Flowrate Increasing • This situation can be best visualized as illustrated in Figure 4 where the tank height is decreasing as a secondary stream removes fluid from the tank while the primary outlet flowrate remains equal to the inlet flowrate. Note that the secondary stream flowrate is time-dependent and will vanish as time increases and a new steady state is established. In this case, removing additional fluid from the tank will only decrease the fluid level in the tank without changing the temperature of the fluid in the tank. The temperature of the fluid in the tank is determined by the rate of heat transfer, Q, that is delivered to the incoming fluid.

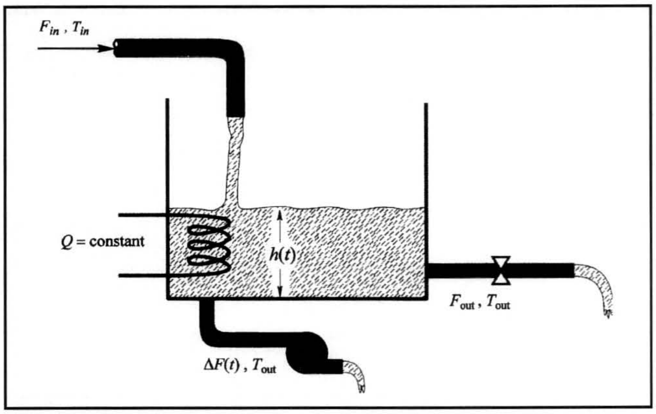

Figure 4. *Increasing the outlet stream flowrate*

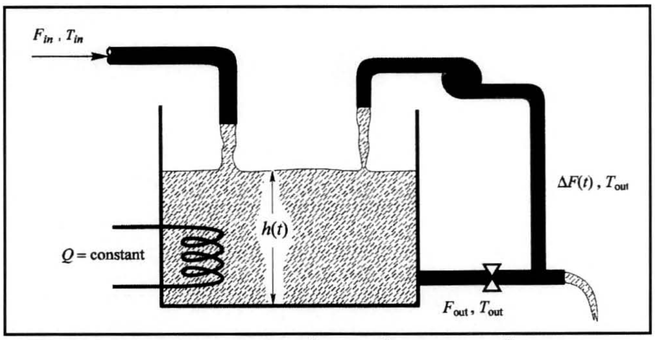

Figure 5. *Decreasing the outlet stream flowrate.*

Case of Outlet Stream Flowrate Decreasing • This situation can be visualized as in Figure 5, where the tank height is increasing as the flow is diverted back to the tank. Again, the diverted stream is time dependent and will vanish as time increases and a new steady state is achieved. Returning some of the outlet stream to the tank will increase the fluid depth in the tank; however, the temperature of the outlet stream is identical to the temperature in the tank and returning a portion of this fluid to the tank will have no influence on the temperature in the tank.

Rather than use the constructions illustrated in Figures 4 and 5 to enhance our intuition, we could simply observe that the energy delivered to the system, Q, is used to raise the temperature of the inlet stream from T_{in} to $\langle T \rangle^{\circ}$, and the disposition of the outlet stream has no influence on this energy-transport process. Thus, the heated fluid in the tank can be disposed of more or less rapidly, giving rise to a change in the fluid depth in the tank, without influencing the temperature in the tank.

CONCLUSIONS

The dynamic behavior of a stirred-tank heater challenges our intuition when mass and energy balances are considered simultaneously. The perfect mixing assumption creates a decoupled dynamic response when changes in the outlet flowrate are considered. We have presented a rigorous model of the process and have offered an explanation as to why we should expect this seemingly counter-intuitive phenomenon to occur.

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