COMPUTER MODELING IN THE UNDERGRADUATE UNIT OPERATIONS LABORATORY

Demonstrating the Quantitative Accuracy of the Bernoulli Equation

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The purpose of this experiment is to demonstrate the predictive capabilities of the Bernoulli equation in determining the time it takes a liquid to drain, under the influence of gravity, from a tank and through an exit pipe, as a function of initial tank charge, exit-pipe diameter, and exit-pipe length. The project is comprised of an experimental component and a modeling component.

In the modeling component, predictions of the efflux time are obtained from several different approximate solutions of the Bernoulli equation; in the experimental component, the flux time for water draining from a tank through various exit pipes is measured. Comparisons between the experimental and theoretical values are then made. The purposes of the comparison are

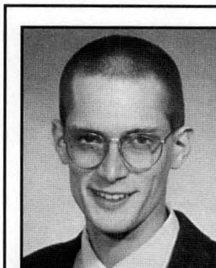
- *To evaluate which terms of the Bernoulli equation are important*
- *To test the limits of applicability of the Bernoulli equation*
- *To demonstrate the value of a rigorous computer modeling*

Descriptions of fluid-flow experiments appear in the literature. For example, Hesketh and Slater described an efflux from a tank experiment where students fit height-versus-time data, assuming there are no pressure losses within the system.[1] In this work, we include head losses due to various friction terms. Hanesian and Perna described an experiment in optimizing pipe diameter with respect to capital and operating costs.[2] A key difference in the latter experiment is that the system was operating at steady state. In the experiment described here, efflux from a tank, there is no steady state, and thus the resulting equations are differential in nature.

EXPERIMENTAL SYSTEM

Our system is situated inside a cylindrical tank (tank radius = R_T) filled with water to height, H. The tank has a cylindrical pipe (pipe radius = R_P) of length L extending from the base of the tank (see Figure 1). The length and the diameter of the stainless steel exit pipe are variables depending on which of the eight available pipes is used. The pipe dimensions are given in Table 1.

The experimental apparatus is intentionally kept as simple as possible. When the students first see the tank and pipes, they frequently smirk and comment that the experiment is too "low-tech" to teach them anything of value, but through this experiment they learn that "The best experiment is the



David Keffer *has been an Assistant Profesor at the University of Tennessee since January, 2000. His research involves the computational description of the behavior of nanoscopically confined fluids. He has transferred the tools of his research—solving algebraic, ordinary, and partial differential equations—to the undergraduate engineering curriculum by integrating modern computer modeling and simulation tools, not only in numerical methods courses but in any engineering course.*

simplest experiment that still has enough guts to demonstrate the underlying physics of the system."[3]

MATHEMATICAL MODEL

The mathematical model used to describe efflux from the tank is based on the mass and mechanical energy balances. If we define our system as the dotted line in Figure 1, and if we stop timing the efflux when the water level reaches H´, then the control volume is always full and we have a mass balance of the form

$$\text{in} = v_T A_T = v_T \pi R_T^2 = \text{out} = v_P A_P v_P \pi R_P^2 \tag{1}$$

assuming an incompressible fluid, where v_T is the flow average velocity in the tank, A_T is the cross-sectional area of the tank, and R_T is the radius of the tank. The subscript P designates analogous variables and parameters of the exit pipe. The average velocity of the fluid in the tank is defined as

$$v_T(t) = \frac{dH}{dt} \tag{2}$$

where t is time. Equation (2) can be substituted into Eq. (1) to yield an expression for the velocity in the pipe

$$v_P = \frac{dH}{dt}\frac{R_T^2}{R_P^2} \tag{3}$$

The mechanical energy balance (Bernoulli equation including friction terms) has the general form

$$\frac{g\Delta z}{g_c} + \frac{\Delta v^2}{2g_c} + \frac{\Delta P}{\rho} + \sum h_f = 0 \tag{4}$$

where g is gravity, $\Delta z = L + H'$, $\Delta v^2 = v_T^2 - v_P^2$, ΔP is the pressure drop, ρ is the density of the fluid, and h_f are the terms contributing to the head loss due to friction.

Again, if we define our system as the dotted line in Figure 1, we have the advantage that the accumulation term within the system over which the material and mechanical energy balance is drawn is zero, since the system is constantly full of liquid. This results in a non-zero pressure drop corresponding to the height of the water in the tank, less H´, the final height at which we stop the experiment.

In this system, we can consider frictional head loss due to the pipe wall, the contraction, and the tank wall

$$\sum h_f = h_{f,\text{pipewall}} + h_{f,\text{contraction}} + h_{f,\text{tankwall}} \tag{5}$$

We define each term in the Bernoulli equation

$$\Delta P = -\frac{\rho g (H - H')}{g_c} \tag{6}$$

The Darcy equation gives the friction head loss for flow in a straight pipe,

$$h_{f,\text{pipewall}} = 4\left(\frac{f_P L}{D_P}\right)\frac{v_P^2}{2g_c} \tag{7}$$

where f_P is a dimensionless friction factor and D_P is the diameter of the pipe.[4] If we assume turbulent flow in the pipe, we can obtain an estimate of the friction factor, f_P, using an empirical relation, known as the Blasius equation, applicable to turbulent flow with Reynolds numbers in the range of $4000 < N_{Re} < 100{,}000$.[4]

$$f_P = \frac{0.0791}{N_{Re,P}^{0.25}} \tag{8}$$

The Blasius equation for a smooth pipe is used because it will allow for an analytical solution to the resulting differential equation. The friction loss due to contraction is given by[5]

$$h_{f,\text{contraction}} = K_c \frac{v_P^2}{2g_c} = 0.5\left(1 - \frac{D_P^2}{D_T^2}\right)\frac{v_P^2}{2g_c} \tag{9}$$

If we assume laminar flow in the tank, the friction loss due to the tank wall is

$$h_{f,\text{tankwall}} = 4f_T\left(\frac{H}{D_T}\right)\frac{v_T^2}{2g_c} = \frac{64}{N_{Re,T}}\left(\frac{H}{D_T}\right)\frac{v_T^2}{2g_c} \tag{10}$$

The assumption of turbulent flow in the pipe and laminar flow in the tank can be verified experimentally. For the diameters and lengths used in this experiment, these assumptions are confirmed.

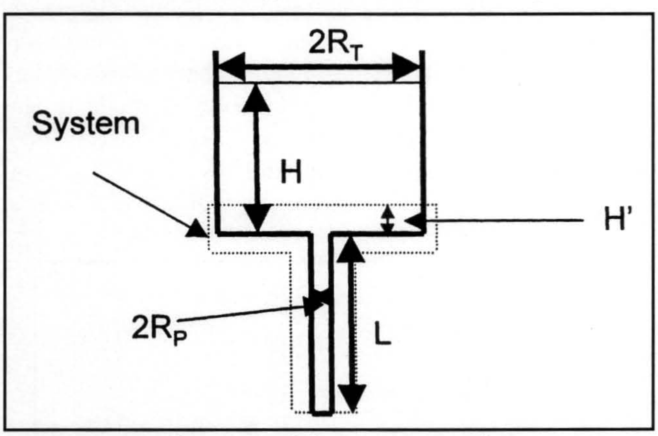


Figure 1. *Schematic of the experimental apparatus.*

TABLE 1
Pipe Dimensions

Length (inches)	*Inside Diameter (inches)*
30	3/16
24	3/16
12	3/16
6	3/16
1	3/16
24	1/8
24	1/4
24	5/16

If we combined Eqs. (1) through (10), we obtain a mechanical energy balance of the form[6]

$$-g(L+H)+\left(\frac{dH}{dt}\right)^{1.75}\left[\frac{2(0.0791)\mu^{0.25}LD_T^{3.5}}{\rho^{0.25}D_P^{4.75}}\right]+$$

$$\left(\frac{dH}{dt}\right)^2\left[\frac{\left(\frac{D_T^4}{D_P^4}\right)-1}{2}\right]+\frac{1}{4}\left(1-\frac{D_P^2}{D_T^2}\right)\left[\frac{D_T^2}{D_P^2}\left(\frac{dH}{dt}\right)\right]^2+\frac{32H\mu}{D_T^2\rho}\left(\frac{dH}{dt}\right)=0 \tag{11}$$

Equation (11) is a first-order nonlinear ordinary differential equation. It has no known analytical solution.

If we rely on our engineering intuition to neglect terms of less significance, however, we might omit the kinetic energy term, the friction loss due to contraction, and the friction loss due to laminar flow in the tank. If we make these three assumptions, we will find that we can obtain an analytical solution to the resulting differential equation

$$t=\left[\frac{2(0.0791)\mu^{0.25}D_T^{3.5}}{g\rho^{0.25}D_P^{4.75}}\right]^{4/7}L\frac{7}{3}\left[\left(1+\frac{H_o}{L}\right)^{3/7}-\left(1+\frac{H(t)}{L}\right)^{3/7}\right] \tag{12}$$

where H_o is the initial height of the water in the tank at time zero. Thus, we can find the time it takes for the water level in the tank to fall to a height, H, from the initial height, H_o. This approximation is what is often used to describe the system in unit operations laboratories solely because it has an analytical solution. We will see in the next section, however, that this approximation gives not only quantitatively but also qualitatively incorrect results.

The more rigorous approach is to numerically solve the ordinary differential equation (ODE) in Eq. (11). We can use a standard numerical ODE-solution technique (*e.g.,* Euler's method or a Runge-Kutta method) if we can arrange the ODE into the form

$$\frac{dH}{dt}=f(H,t) \tag{13}$$

Equation (11) cannot be put in this form. Therefore, we cannot easily solve for the velocity in the tank, DH/dt, at every Euler or Runge-Kutta time step as is required by those algorithms. But for any given time, t, for which we know the height, H, we can obtain the numerical value of the tank velocity by using a technique to solve a single nonlinear algebraic equation, such as the Newton-Raphson method. Combining the Newton-Raphson and Runge-Kutta methods is a relatively simple algorithm to implement and involves nesting the iterative algebraic equation solver inside the routine that obtains the tank velocity for the ODE solver. For the undergraduates in the unit operations laboratory, we provide just such a routine, written for MATLAB.[6] The students are familiar individually with the Runge-Kutta and Newton-Raphson techniques and the majority of them directly comprehend the combination of the two methods.

We have integrated the modeling component of this experiment with the curriculum-wide "Web Resource for the Development of Modern Engineering Problem-Solving Skills" instituted in the Department of Chemical Engineering at the University of Tennessee.[7] This web resource acts as a stand-alone self-teaching module that students at any level in the program—from sophomores to graduate students—can access to obtain the basic algorithms to solve systems of linear algebraic equations, systems of nonlinear algebraic equations, systems of ordinary differential

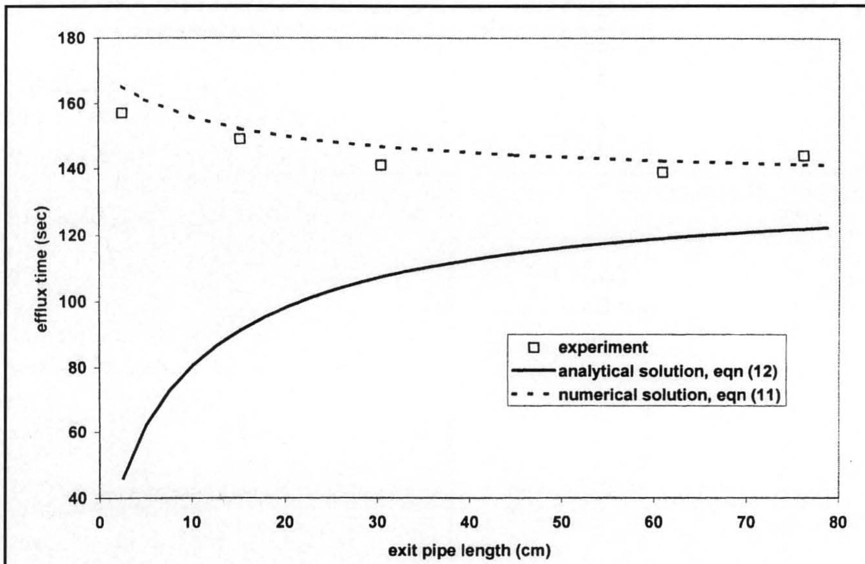


***Figure 2.** Efflux time as a function of exit pipe length for the experimental case, the approximation to the mechanical energy balance with an analytical solution (Eq. 12), and for more complete mechanical energy balance, solved numerically (Eq. 11). The data are for water at 85°F draining from a six-inch diameter baffled tank from an initial height of 11 in. to a final height of 2 in. through a pipe with a nominal diameter of 3/16 in.*

equations, numerical integration, and linear regression and analysis of variance.

EXPERIMENTAL RESULTS

In the lab the students examine the effects on efflux time of the initial water charge, the exit-pipe diameter, and the exit-pipe length. Here, we limit ourselves to the effect of the exit-pipe length. In Figure 2 we plot the flux time versus exit-pipe length for the experimental case, for the approximation to the mechanical energy balance with an analytical solution (Eq. 12), and for the complete mechanical energy balance, solved numerically (Eq. 11). The data are for water at 85°F draining from a six-inch diameter baffled tank from an initial height of 11 in. to a final height of 2 in. through a pipe with nominal diameter of 3/16 in. The water density and viscosity were obtained from the literature.[8]

At short pipe lengths, we see that the experimental efflux time decreases with increasing pipe length, because gravity and the hydrostatic pressure term in Eq. (11) create a driving force for flow proportional to (L+H). As we increase L, the driving force increases and the tank drains faster. In contrast, at longer pipe lengths, the experimental efflux time increases with increasing pipe length, because we have reached a point where skin friction due to the pipe wall is the dominating factor.

The approximation to the Bernoulli equation that has an analytical solution (Eq. 12) fails to model this behavior both qualitatively and quantitatively. The trend for Eq. (12) is a monotonic increase in efflux time with increasing pipe length. The average relative error of Eq. (12) with respect to the experimental data is 32.6%.

The more complete Bernoulli equation in Eq. (11) models the experiment both qualitativelty and quantitatively. The average relative error of Eq. (11) with respect to the experimental data is 3.1%.

Plots have also been generated regarding the dependence of efflux time on pipe diameter and initial water height. Both the analytical solution (Eq. 12) and the numerical solution to Eq. (12) model the behavior qualitatively, namely that efflux time decreases as pipe diameter increases or initial water height decreases. But as was the case with the pipe length, the quantitative agreement is substantially better using Eq. (11).

CONCEPTUAL LESSONS OF THE EXPERIMENT

After the students have collected the experimental data in the laboratory, they take the data to the computer lab and model it using both Eqs. (11) and (12). Additionally, they look at variant models, adding one term at a time—kinetic energy, friction due to contraction, and friction due to the laminar flow in the tank wall. Adding the terms individually allows the student to determine the effect of each term in the mechanical energy balance on the efflux time.

The students can also explore the comparison of experiment and theory in terms of error analysis. For example, they can calculate the Reynolds number at each experimental data point and show that for any given theoretical model the accuracy decreases as the Reynolds number drops and reaches the lower limit of applicability of the expression used for the turbulent friction factor.

Finally, the students (primarily juniors) obtain a first-hand demonstration of the quantitative accuracy of the Bernoulli equation. The experience helps them understand the significance, validity, and limitations of the otherwise abstract mathematical expressions with which they are presented in classroom lectures on fluid flow.

CONCLUSIONS

In this work we have described a very simple efflux from a tank experiment, of the sort commonly employed in undergraduate unit operations laboratory courses. We have shown that relying only on a simplified analytical solution to the Bernoulli equation not only fails to quantitatively model the experimental results but also qualitatively fails to capture the correct trends. We have provided a more complete mechanical energy balance, outlined its numerical solution, and shown that it both qualitatively and quantitatively models the experiment.

The inclusion of a computer simulation in the experiment allows the students to demonstrate for themselves the consequences of over-simplified engineering approximations and the value of a rigorous mathematical model.

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