A NOTE ON STABILITY ANALYSIS USING BODE PLOTS

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The Bode plot is an important tool for stability analysis of closed-loop systems. It is based on calculating the amplitude and phase angle for the transfer function

$$G_{OL}(s) = G_C(s)G_P(s) \tag{1}$$

for

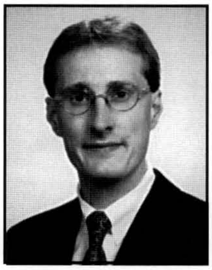
$$s = j\omega$$

where $G_C(s)$ is the controller and $G_P(s)$ is the process. The Bode stability criterion presented in most process control textbooks is a sufficient, but not necessary, condition for instability of a closed-loop process.[1-4] Therefore, it is not possible to use this criterion to make definitive statements about the stability of a given process.

Other textbooks[5,6] state that this sufficient condition is a necessary condition as well. That statement is not correct, as will be demonstrated in the following examples. In another text[7] the criterion is formulated as a necessary condition for stability, but no definite statements can be made based on a necessary condition alone.

Often, some statements are added for clarification,[1-6,8] *e.g.*, "...the Bode stability criterion only applies to systems that cross $\phi = -180°$ once, where ϕ is the phase shift of the transfer function $G_C(s)G_P(s)$. For multiple crossings one must use the Nyquist criterion."[3]

It can be shown that the above statements about the Bode stability criterion are not complete. For example, a system can cross the -180° phase angle line only once, have an amplitude ratio of less than one at the corresponding frequency, and still be unstable. This is because $G_C(s)G_P(s)$ can have an amplitude ratio greater than unity for frequencies where $\phi = -180° - n*360°$, where n is an integer. These conditions can occur when the process includes time delays, as shown in the following example.



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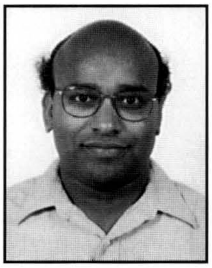

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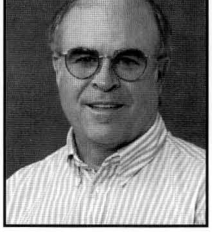

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EXAMPLE 1

A reboiler process with a proportional-derivative (PD) Controller

A process transfer function with inverse response and integrating action as seen in some reboilers is to be controlled by a PD controller with a pre-filter.

$$G_P(s) = \frac{-s+1}{s(\tau s+1)} e^{-\theta s} \qquad (2)$$

$$G_C(s) = K_C \frac{\tau_D s+1}{\alpha \tau_D s+1} \qquad (3)$$

The reason that a PD controller is chosen over a PID is that the process transfer function already contains a pole at the origin. Therefore, the controller for this process should not include integral action for impulse or step inputs in the setpoint or disturbances.

The parameters are chosen to be

τ = 0.1 min
θ = 0.4 min
K_c = 0.2
τ_D = 1 min
α = 0.05

The resulting Bode plot for the open-loop system is given in Figure 1. The phase crossover frequency ($\omega_{c180°}$) is defined to be the frequency at which the open-loop phase angle is -180°. Furthermore, the gain crossover frequencies (ω_g) are defined to be the frequencies at which the open-loop amplitude ratio is equal to unity and $\omega_{c540°}$ corresponds to the frequency where the phase angle crosses -540°.

The amplitude ratio corresponding to a phase lag of -180° is 0.6. One could reach the following false conclusions from the Bode stability criterion:

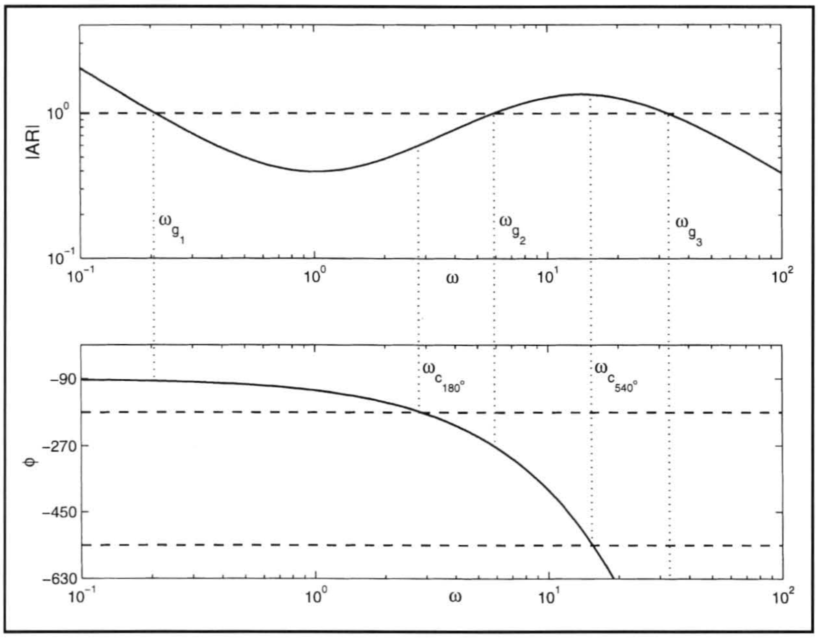

Figure 1. *Bode plot of the reboiler process.*

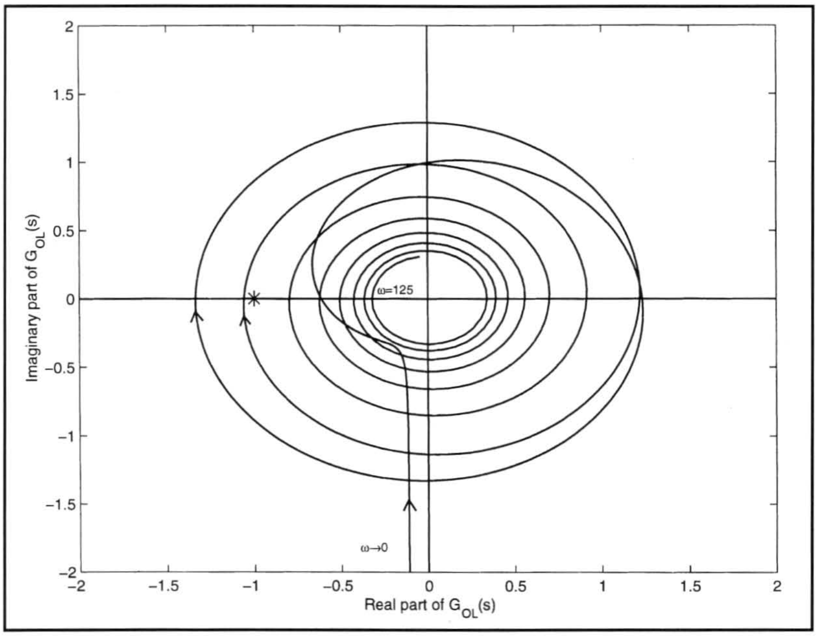

Figure 2. *Nyquist plot of the reboiler process.*

- ▶ *The system is stable.*
- ▶ *The gain margin is 1.67. The controller gain can be increased by 67% without making the system unstable.*

But the amplitude ratio corresponding to a phase angle of -540° is 2.0 and thus greater than unity. Therefore, we conclude that the closed-loop system is unstable. Increasing the controller gain makes the system even more unstable. Instead, a reduction of the gain by 50% will result in a stable closed-loop system.

These conclusions can be validated by analyzing the Nyquist plot in Figure 2. It is apparent that this system is unstable due to the fact that the curve shown encircles the point (-1,0) twice in a clockwise direction. When this system is implemented in MATLAB and simulated, the finding from the Nyquist plot is confirmed.

Another commonly found statement about the Bode stability criterion is that it cannot be used if the frequency response of the open-loop system exhibits "nonmonotonic phase angles or amplitude ratios at frequencies higher than the first phase crossing of -180°."[8] Although this statement applies to many cases and would exclude the above example, it can lead to false conclusions. For example, the amplitude ratio of a system can be monotonically increasing for a PID controller after the notch frequency,[5] while the phase angle is constantly decreasing due to a time delay in the process. It is possible to construct a case where the notch frequency of the system has a phase lag of less than 180° and the corresponding amplitude ratio is less than unity. Although this system behaves monotonically in both phase and amplitude ratio after its phase crossover frequency, further analysis is required to determine the stability of the system. Example 2 illustrates this point.

EXAMPLE 2

Control of a time-delay process with an electronic proportional-integral-derivative (PID) controller

Assume a process that consists of a pure time delay and is controlled by a PID controller.

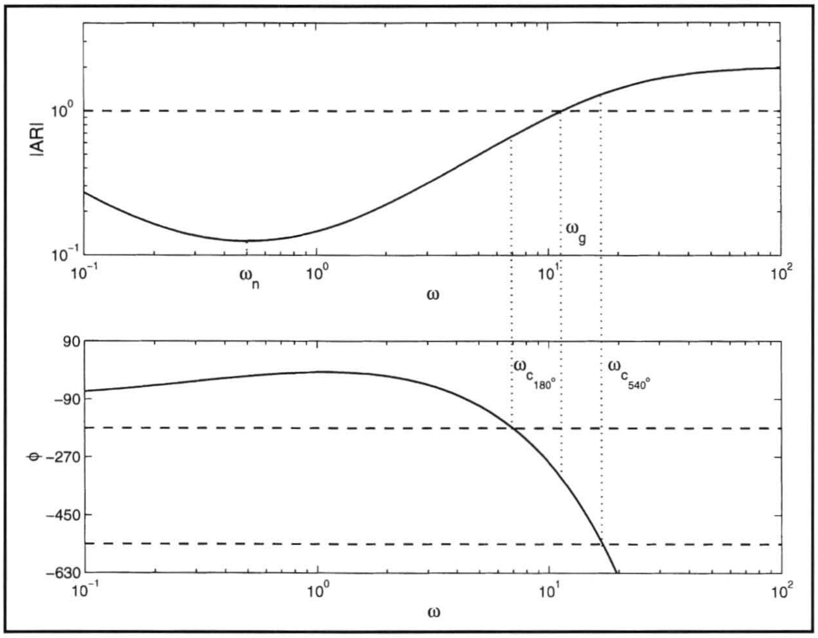

Figure 3. *Bode plot of the time-delay process.*

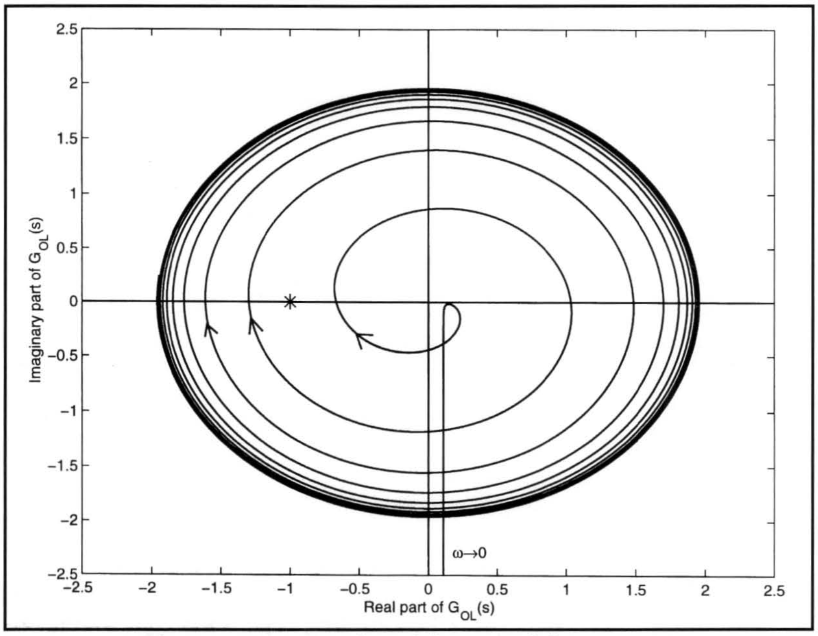

Figure 4. *Nyquist plot of the time-delay process.*

$$G_P(s) = e^{-\theta s} \quad (4)$$

$$G_C(s) = K_C \frac{\tau_I s + 1}{\tau_I s} \frac{\tau_D s + 1}{\alpha \tau_D s + 1} \quad (5)$$

The parameters are given by

θ = 0.6 min
K_C = 0.1
τ_I = 4 min
τ_D = 1 min
α = 0.05

Figures 3 and 4 show the Bode and Nyquist plots of the open-loop transfer function of this process.

The notch frequency (ω_n) for this PID controller is 0.5 min^{-1} and the amplitude ratio of the open-loop process is monotonically increasing at higher frequencies. The phase crossover frequency is located at $\omega_{c180^\circ} = 0.7\ min^{-1}$ and at higher frequencies, both the amplitude ratio and the phase angle are monotonic. It can be concluded from the Nyquist diagram that this system is unstable, since (-1,0) is encircled an infinite number of times in a clockwise direction. This result has been confirmed in simulations. If the Bode stability criterion in any of the above mentioned forms is used to determine stability of the system, however, it would lead to the wrong conclusion.

The foregoing examples indicate that it is not possible to formulate a Bode stability criterion that is simple to use and applicable to all possible cases at the same time. Therefore we conclude:

- ► A system should only be analyzed for stability using the Bode plot, if it has at most one phase crossover frequency. Additionally, if it has only one gain crossover frequency and the amplitude ratio as well as the phase angle are decreasing at the gain crossover and afterward, then the gain and phase margins can be calculated in a way found in control textbooks.
- ► A system that has only one phase crossover frequency but multiple gain crossover frequencies is stable if the amplitude ratios, corresponding to frequencies where $\phi = -180^\circ - n*360^\circ$, are all less than unity and the open-loop system is stable. The gain margin is calculated from the crossover frequency or a frequency corresponding to a larger n, whichever exhibits the largest amplitude ratio.
- ► If the Bode plot information is inconclusive, the Nyquist stability criterion should be applied for stability analysis of closed-loop systems.

From these conclusions we propose a

Revised Bode Stability Criterion

> A closed-loop system is stable if the open-loop system is stable and the frequency response of the open-loop transfer function has an amplitude ratio of less than unity at all frequencies corresponding to $\phi = -180^\circ - n*360^\circ$, where $n = 0,1,2,\ldots,\infty$.

The proof of the revised Bode stability criterion follows directly from the Nyquist criterion. When this definition of a stability criterion is recast in a form for use in a Nyquist diagram, the resulting set of closed-loop stable systems is given by the curves that do not cross the real axis to the left of (-1,0) and are open-loop stable. Therefore, all these curves do not encircle (-1,0) in either direction, and this set is a subset of all stable closed-loop systems described by the Nyquist stability criterion.

This revised stability criterion is a sufficiency condition for stability. It is not a necessary condition, since a system can have multiple phase crossover frequencies (some of them with amplitude ratios larger than unity) and still be stable. If a case arises that is not covered by the revised criterion, then the Nyquist stability criterion should be used for stability analysis.

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