SIMPLE USES OF LAPLACE TRANSFORMS
in Transient Transport Problems

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With the ever-increasing use of computers in the solution of partial differential equations, it may be advantageous to emphasize those analytical tools that are easy and quick to implement. Referring specifically to the Parabolic Partial Differential Equations (PPDE) arising in the transient problems of transport phenomena, the method of Laplace Transforms can be, by far, the most expedient technique in solving them. Carslaw and Jaeger have provided a thorough treatment of the subject in their texts, *(e.g.,* the classic *Conduction of Heat in Solids*[1]) and the first book of applied mathematics in chemical engineering covered the topic in reasonable detail.[2] Yet this long-known technique has received generally little attention in chemical engineering textbooks and classrooms.

This article attempts to show how a few classic transient-transport problems are solved through Laplace Transforms—more quickly than through techniques that are generally used in chemical engineering books. Also, problems, which may be avoided in graduate courses and texts because of the perceived length of time needed to cover them, are shown here to be solvable in little time and few mathematical steps. In the past, in inverting Laplace expressions using the theory of residues, an unattractive feature of the technique may have been the attempt to do the inversions by starting from first principles. Whereas there are certainly problems where using the theory of residues is necessary, the approach of this article is to present methodologies that make use of the well-known Laplace Transform properties together with readily available tables of Laplace inverses. Since all relevant engineering and mathematics texts discuss the properties of Laplace Transforms, reference will be made here only to Spiegel's Mathematical Handbook (SMH).[3] For the sake of limiting the reproduction of standard equations, references to SMH and BSL (Bird, Stewart, and Lightfoot)[4] will be made using the equation numbers as they appear in those texts.

DEFINITION, NOTATION AND BASIC PROPERTIES

The Laplace Transform of a function f(x,t) with respect to time, t, is denoted and defined as

(the definition)

$$\mathcal{L}\{f(x,t)\} = \bar{f}(x,s) = \int_0^\infty f(x,t)e^{-st}\,dt \tag{1}$$

where s is the Laplace variable.

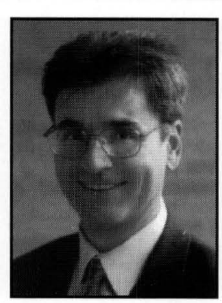
The operation $\mathcal{L}: t \to s$ (or $\mathcal{L}: \tau \to s$) signifies Laplace transformation of a PDE and its boundary conditions from the real time domain, t or τ, to the Laplace domain, s, while $\mathcal{L}^{-1}: s \to t$ (or τ) is the inversion of an expression in the Laplace domain, s, back to the real time domain, t or τ. Powerful properties for inversion, used in the following series of equations, will be referred to by the names in the parentheses that precede each equation

(translation property)

$$\mathcal{L}^{-1}\{\bar{f}(x,s+a)\} = e^{-at}\,\mathcal{L}^{-1}\{\bar{f}(x,s)\} = e^{-at}f(x,t) \tag{2}$$

(derivative property)

$$\mathcal{L}\left\{\frac{\partial f(x,t)}{\partial t}\right\} = s\bar{f}(x,s) - f(x,t=0) \tag{3}$$



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(over – s property)

$$\mathfrak{L}^{-1}\left\{\frac{\bar{f}(x,s)}{s}\right\} = \int_0^\tau \mathfrak{L}^{-1}\{\bar{f}(x,s)\}dt = \int_0^\tau f(x,t)dt \qquad (4)$$

(convolution property)

$$\mathfrak{L}^{-1}\{\bar{f}(x,s)\bar{g}(x,s)\} = \int_0^\tau f(x,t)g(x,\tau - t)dt \qquad (5)$$

For the sake of brevity, when the arguments of a function are evident they may not always be indicated, *e.g.*, f(x,t) may appear simply as f, and $\bar{f}(x,s)$ may be represented as $\bar{f}$ or $\bar{f}(s)$.

EXAMPLES FROM BSL

In BSL the only problem solved by Laplace Transforms is Example 11.1-3 (cooling of a sphere in contact with a well-stirred fluid), where the theory of residues is needed for inversion. In this article we will deal only with problems that do not necessitate use of residues, but rely simply on the tabulated inversions in SMH.

BSL Examples 4.1-1 and 11.1-1

These examples address the transient diffusion of momentum and heat in a semi-infinite medium. For the flow near a wall suddenly set in motion, this is described by the PDE

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2} \qquad (6)$$

with conditions

$$v(y,t=0)=0 \qquad v(y=0,t)=V \qquad v(y=\infty,t)=0$$

The method of solution of this problem in BSL is similarity transformation (combination of variables), the didactically detailed treatment of which is given by Whitaker.[5]

Using Laplace Transforms to solve the above equation, $\mathfrak{L}:\tau \to s$ of Eq. 6 with simultaneous use of the initial condition yields

$$s\bar{v} = \nu \frac{d^2\bar{v}}{dy^2} \qquad (7)$$

with conditions

$$\bar{v}(y=0,s)=\frac{V}{s} \qquad \bar{v}(y=\infty,s)=0$$

which has s-domain solution

$$\bar{v} = V\frac{e^{-\sqrt{\frac{s}{\nu}}y}}{s} \qquad (8)$$

Applying SMH 32.111 we may invert, $\mathfrak{L}^{-1}:s \to t$, to obtain BSL's Eq. 4.1-13.

Notice that the Laplace Transform may as easily be used to solve more difficult variations of this problem, such as those involving a nonconstant velocity for the wall. Accordingly, if the wall is set in motion with a linearly increasing velocity so that the relevant boundary condition assumes the form v(y=0,t)=Bt for $0 < t < t_B$, the Laplace transformed condition will be

$$\bar{v}(y=0,s)=\frac{B}{s^2} \qquad (9)$$

and

$$\bar{v} = B\frac{e^{-\sqrt{\frac{s}{\nu}}y}}{s^2} \qquad (10)$$

which is inverted either through SMH 32.113 or by using the "over-s property" on SMH 32.111.

BSL Example 4.1-2

In this problem, the dimensionless transient velocity profile in a tube, $\varphi(\xi,\tau)$, is the solution of

$$\frac{\partial\varphi}{\partial\tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\varphi}{\partial\xi}\right) \qquad (11)$$

where ξ and τ are the dimensionless radial position and time respectively. The initial boundary conditions for this problem are

$$\varphi(\xi,\tau=0)=0 \qquad \varphi(\xi=0,\tau)=\text{finite} \qquad \varphi(\xi=1,\tau)=0$$

The solution of this problem in BSL follows the original long and tedious "separation-of-variables" Szymanski treatment. Using Laplace Transforms instead, the problem is solved in a few simple steps as follows:

$\mathfrak{L}:t \to s$ Eq. 11 gives

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\bar{\varphi}}{d\xi}\right) - s\bar{\varphi} = -\frac{4}{s} \qquad (12)$$

where $\bar{\varphi}$ is the Laplace transformed dimensionless velocity profile. Eq. 12 is an ordinary Bessel differential equation with a solution

$$\bar{\varphi} = C_1 I_0\left(\sqrt{s}\xi\right) + C_2 K_0\left(\sqrt{s}\xi\right) + \frac{4}{s^2} \qquad (13)$$

The integration constants C_1 and C_2 are found as C_2=0 from the center condition, $\bar{\varphi}(\xi=0)$=finite, whereas the wall condition, $\bar{\varphi}(\xi=1)=0$, gives

$$C_1 = \frac{-4}{\left\{s^2 I_0\left(\sqrt{s}\right)\right\}} \qquad (14)$$

so that

$$\bar{\varphi}=\frac{4}{s^2}-\frac{4I_0\left(\sqrt{s}\xi\right)}{s^2 I_0\left(\sqrt{s}\right)} \tag{15}$$

Using the relationship between the Bessel function of the first kind, J_0, and its modified counterpart, I_0, Eq. 15 may be written as

$$\bar{\varphi}=\frac{4}{s^2}-\frac{4J_0\left(i\sqrt{s}\xi\right)}{s^2 J_0\left(i\sqrt{s}\right)} \tag{16}$$

In inverting Eq. 16, $\mathfrak{L}^{-1}: s \to \tau$, use is made of the tabulated inversion 32.157 in SMH for the second part of Eq. 16 and φ is found as in BSL Eq. 4.1-40. Notice that φ does not have to be the dimensionless velocity as defined in BSL, but it may be the "deviation" dimensionless profile, from some original Hagen-Poiseuille steady-state flow.[6]

UNSTEADY DIFFUSION-REACTION IN CATALYSTS

Whereas the steady-state diffusion with first-order reaction in spherical catalysts is a classic example covered in both Transport Phenomena and Reactor Design texts, its equally important unsteady-state version is not. This is due, in the opinion of this author, to the many tedious and lengthy steps required by the familiar separation-of-variables solution. In Papadopoulos, [6] it was shown that when a spherical catalyst particle is originally operating at steady state with a surface boundary condition $C(\xi=1)=C_1$, and suddenly the surface concentration is changed to C_2, the PPDE is

$$\frac{\partial\Gamma}{\partial\tau}=\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Gamma}{\partial\xi}\right)-\beta^2\Gamma \tag{17}$$

where $\xi = r/R$

R = radius of the sphere

$\tau = t\mathfrak{D}/R^2$

t = time

$\mathfrak{D}$ = Diffusivity

$\beta^2 = k_1R^2/\mathfrak{D}$

k_1 = first-order reaction rate constant

Γ = deviation concentration from an original steady-state concentration, $C_{ss1}(\xi)=C_1 \sinh \beta\xi/[\xi \sinh \beta]$, defined as $\Gamma = C(\xi,\tau)-C_{ss1}(\xi)$

The conditions for solving Eq. 17 are

$$\Gamma(\tau=0)=0 \quad \Gamma(\xi=0)=\text{finite for all }\tau \quad \Gamma(\xi=1)=\Gamma_0$$

with $\Gamma_0=C_2-C_1$.

The PPDE is solved by first $\mathfrak{L}: \tau \to s$ Eq. 17, using its initial condition

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\bar{\Gamma}}{d\xi}\right)-\left(s+\beta^2\right)\bar{\Gamma}=0 \tag{18}$$

Solving this ordinary differential equation (ODE) gives

$$\bar{\Gamma}=K_1'\frac{\sinh\sqrt{s+\beta^2}\,\xi}{\xi}+K_2'\frac{\cosh\sqrt{s+\beta^2}\,\xi}{\xi} \tag{19}$$

The center condition causes integration constant K_2' to be zero, while the Laplace transformed surface condition,

$$\bar{\Gamma}(\xi=1)=\frac{\Gamma_0}{s}$$

produces

$$K_1'=\frac{\Gamma_0}{s\sinh\sqrt{s+\beta^2}}$$

and

$$\frac{\bar{\Gamma}}{\Gamma_0}=\frac{\sinh\sqrt{s+\beta^2}\,\xi}{\xi\, s\sinh\sqrt{s+\beta^2}} \tag{20}$$

which, for the purpose of showing the steps of inversion, is written as

$$\frac{\bar{\Gamma}}{\Gamma_0}\xi=\frac{\bar{f}\left(s+\beta^2\right)}{s} \tag{21}$$

with

$$\bar{f}(s)=\frac{\sinh\sqrt{s}\,\xi}{\sinh\sqrt{s}} \tag{22}$$

Whereas the inverse of Eq. 20 may not be readily available in most lists of Laplace Transforms, the inverse of Eq. 22, $\mathfrak{L}^{-1}\{\bar{f}(s)\}$, is given by SMH 32.148. Using this together with the "over-s property"

$$\begin{aligned}
\frac{\Gamma(\xi,\tau)}{\Gamma_0}&=\frac{1}{\xi}\int_0^\tau \mathfrak{L}^{-1}\{\bar{f}(s)\}e^{-\beta^2 t}\,dt\\
&=\frac{1}{\xi}\int_0^\tau 2\pi\sum_{n=1}^{\infty}(-1)^n n e^{-n^2\pi^2 t}\sin(n\pi\xi)e^{-\beta^2 t}\,dt\\
&=\frac{2\pi}{\xi}\sum_{n=1}^{\infty}(-1)^n n\sin(n\pi\xi)\int_0^\tau e^{-\left(n^2\pi^2+\beta^2\right)t}\,dt\\
&=\frac{2\pi}{\xi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi\xi)}{n^2\pi^2+\beta^2}\\
&\quad-\frac{2\pi}{\xi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi\xi)}{n^2\pi^2+\beta^2}e^{-\left(n^2\pi^2+\beta^2\right)\tau}
\end{aligned} \tag{23}$$

Notice that the first term in the final expression of Eq. 23 is the final steady state, since as $t \to \infty$ the second term vanishes. Since the final steady state is known independently to be

$$\frac{\Gamma_{ss2}}{\Gamma_0} = \frac{\sinh \beta\xi}{\xi \sinh \beta} \tag{24}$$

the final solution may also be written as

$$\frac{\Gamma(\xi,\tau)}{\Gamma_0} = \frac{\sinh \beta\xi}{\xi \sinh \beta} - \frac{2\pi}{\xi} \sum_{n=1}^{\infty} \frac{(-1)^n n \sin(n\pi\xi)}{n^2\pi^2 + \beta^2} e^{-(n^2\pi^2+\beta^2)\tau} \tag{25}$$

and a side result of this exercise is the mathematical identity

$$2\pi \sum_{n=1}^{\infty} \frac{(-1)^n n \sin(n\pi\xi)}{n^2\pi^2 + \beta^2} = \frac{\sinh \beta\xi}{\sinh \beta} \tag{26}$$

PROBLEMS WITH PERIODIC BOUNDARY CONDITIONS

Reaction-Diffusion in a Catalyst

Unsteady-state problems may possess periodicity in a boundary condition. An example is the unsteady catalyst diffusion-reaction considered in the previous section, with the difference that the surface concentration imposed at t=0 is no longer C_2, but a sinusoid around C_1, *i.e.*, $C_1 + C_1 \sin(\omega\tau)$. The PPDE is still given by Eq. 17 while the surface boundary condition now takes the form $\Gamma(\xi=1,\tau) = \Gamma_0 \sin \omega\tau$, where Γ_0 is the original surface concentration, C_1. In solving this periodic-boundary problem, Eqs. 17-19 are unchanged, K_2' is zero as before, though the other integration constant is found by using the new surface-concentration condition in the Laplace domain,

$$\bar{\Gamma}(\xi=1) = \frac{\Gamma_0 \omega}{\left(s^2+\omega^2\right)} \tag{27}$$

which leads to

$$K_1' = \frac{\Gamma_0 \omega}{\left\{\left(s^2+\omega^2\right) \sinh \sqrt{s+\beta^2}\right\}} \tag{28}$$

and

$$\frac{\bar{\Gamma}}{\Gamma_0} = \frac{\omega \sinh \sqrt{s+\beta^2}\, \xi}{\xi\left(s^2+\omega^2\right) \sinh \sqrt{s+\beta^2}} \tag{29}$$

which may be written as

$$\frac{\bar{\Gamma}}{\Gamma_0}\frac{\xi}{\omega} = \frac{\sinh \sqrt{s+\beta^2}\, \xi}{\left(s^2+\omega^2\right) \sinh \sqrt{s+\beta^2}} = \bar{h}(s)\bar{f}\left(s+\beta^2\right) \tag{30}$$

where $\bar{f}(s)$ is still given by Eq. 22, and

$$\bar{h}(s) = \frac{1}{\left(s^2+\omega^2\right)} \tag{31}$$

Since we may readily have the inverses of $h(s)$, SMH 32.32, and of $\bar{f}(s)$, SMH 32.148, Eq. 29 may be inverted by invoking the "convolution property" and "translation property" as follows:

$$\frac{\Gamma(\xi,\tau)}{\Gamma_0} = \frac{\omega}{\xi} \int_0^{\tau} \mathfrak{L}^{-1}\left\{\bar{h}(s)\right\}\Big|_t \mathfrak{L}^{-1}\left\{\bar{f}\left(s+\beta^2\right)\right\}\Big|_{\tau-t} dt$$

$$= \frac{\omega}{\xi} \int_0^{\tau} \frac{\sin \omega t}{\omega} 2\pi \sum_{n=1}^{\infty} (-1)^n n e^{-n^2\pi^2(\tau-t)} \sin(n\pi\xi) e^{-\beta^2(\tau-t)} dt$$

$$= \frac{2\pi}{\xi} \sum_{n=1}^{\infty} (-1)^n n \sin(n\pi\xi) e^{-(n^2\pi^2+\beta^2)\tau} \int_0^{\tau} \sin \omega t\, e^{(n^2\pi^2+\beta^2)t} dt$$

$$= \frac{2\pi}{\xi} \sum_{n=1}^{\infty} (-1)^n n \sin(n\pi\xi) e^{-(n^2\pi^2+\beta^2)\tau}$$

$$\left\{ \frac{e^{(n^2\pi^2+\beta^2)\tau}\left[\left(n^2\pi^2+\beta^2\right)\sin \omega\tau - \omega \cos \omega\tau\right]}{\left(n^2\pi^2+\beta^2\right)^2+\omega^2} + \frac{\omega}{\left(n^2\pi^2+\beta^2\right)^2+\omega^2} \right\} \tag{32}$$

finally leading to

$$\frac{\Gamma(\xi,\tau)}{\Gamma_0} = \frac{2\pi}{\xi} \sum_{n=1}^{\infty} (-1)^n n \sin(n\pi\xi) \left\{ \frac{\left[\left(n^2\pi^2+\beta^2\right)\sin \omega\tau - \omega \cos \omega\tau\right]}{\left(n^2\pi^2+\beta^2\right)^2+\omega^2} \right\}$$

$$+ \frac{2\pi}{\xi} \sum_{n=1}^{\infty} (-1)^n n \sin(n\pi\xi) \left\{ \frac{\omega e^{-(n^2\pi^2+\beta^2)\tau}}{\left(n^2\pi^2+\beta^2\right)^2+\omega^2} \right\} \tag{33}$$

PROBLEMS WITH PERIODIC DRIVING FORCE

Pulsatile Flow

The classic pulsatile-flow-in-a-tube problem is one where the periodicity exists in the driving force instead of the boundary.[7] The way the problem is set up here is by considering the situation where originally a fluid is under steady-state Hagen-Poiseuille flow with a pressure gradient

$$\frac{(P_0 - P_L)}{L}$$

when suddenly the gradient starts fluctuating around the original value as

$$\frac{(P_0 - P_L)}{L} + \frac{P_0 - P_L}{L} \sin \omega\tau \tag{34}$$

Making reference to Papadopoulos,[6] it can easily be shown that the differential equation and boundary conditions are

$$\frac{\partial \varphi}{\partial \tau} = 4 \sin \omega\tau + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \varphi}{\partial \xi}\right) \tag{35}$$

and

$$\varphi(\xi, \tau = 0) = 0 \quad \varphi(\xi = 0, \tau) = \text{finite} \quad \varphi(\xi = 1, \tau) = 0$$

where φ is the deviation velocity from the original steady state[6] and is non-dimensionalized as shown in BSL Eq. 4.1-18. Laplace transformation of Eq. 35, $\mathfrak{L}: t \to s$, with the aid of initial condition, gives

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d\bar{\varphi}}{d\xi}\right) - s\bar{\varphi} = -\frac{4\omega}{s^2 + \omega^2} \tag{36}$$

Solving the ODE above, using the center and surface conditions, leads to

$$\bar{\varphi} = \frac{4\omega}{s(s^2 + \omega^2)} - \frac{4\omega}{(s^2 + \omega^2)}\frac{I_0(\sqrt{s}\,\xi)}{sI_0(\sqrt{s})} = 4\omega\left\{\frac{\bar{h}(s)}{s} - \bar{h}(s)\bar{g}(s)\right\} \tag{37}$$

where

$$\bar{h}(s) = \frac{1}{(s^2 + \omega^2)} \tag{38}$$

and

$$\bar{g}(s) = \frac{I_0(\sqrt{s}\,\xi)}{sI_0(\sqrt{s})} = \frac{J_0(i\sqrt{s}\,\xi)}{sJ_0(i\sqrt{s})} \tag{39}$$

whose inverses are found in SMH 32.32 and 32.156. To invert Eq. 37, $\mathfrak{L}^{-1}: s \to \tau$, use the "over-s" and "convolution" properties as follows:

$$\varphi(\xi, \tau) =$$

$$4\omega\left\{\int_0^\tau \frac{\sin \omega t}{\omega}\,dt - \int_0^\tau \frac{\sin \omega t}{\omega}\left[1 - 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n \xi)}{\lambda_n J_1(\lambda_n)}e^{-\lambda_n^2(\tau - t)}\right]dt\right\} \tag{40}$$

where λ_n are the positive roots of the Bessel function J_0. Noticing, in the expression above, the cancellation of the first integral and the first term of the second integral, the steps summarized below lead to the following solution whose second term disappears as time goes to infinity

$$\varphi(\xi,\tau) = 4\omega\left\{\int_0^\tau \frac{\sin\omega t}{\omega}\left[2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}e^{-\lambda_n^2(\tau-t)}\right]dt\right\} = 8\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}e^{-\lambda_n^2\tau}\int_0^\tau \sin\omega t\, e^{\lambda_n^2 \tau}\,dt$$

$$= 8\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}e^{-\lambda_n^2\tau}\left\{\frac{e^{\lambda_n^2\tau}\left(\lambda_n^2 \sin\omega\tau - \omega\cos\omega\tau\right)}{\lambda_n^4 + \omega^2} + \frac{\omega}{\lambda_n^4 + \omega^2}\right\} = 8\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}\frac{\lambda_n^2 \sin\omega\tau - \omega\cos\omega\tau}{\lambda_n^4 + \omega^2}$$

$$+8\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}\frac{\omega}{\lambda_n^4 + \omega^2}e^{-\lambda_n^2\tau} \tag{41}$$

CONCLUSIONS

Whereas Laplace Transforms have been part of every chemical engineer's training at some point in their undergraduate and graduate years, their usefulness is not exploited adequately in textbook treatments of transport-phenomena PPDEs. This article has presented simple methodologies for solving such equations in efficient ways (*e.g.,* BSL examples 4.1-1, 11.1-1 and BSL 4.1-2). It has also shown that familiarity with the technique may allow a quick coverage of a wide class of important problems in graduate courses—*e.g.*, unsteady diffusion-reaction and problems with periodic boundaries or driving forces—whose solution by other techniques may render them too lengthy to merit class coverage.

As a word of caution, this treatment is recommended mostly for graduate transport phenomena courses where there is little emphasis on transient transport. On the other hand, if there exists a two-semester transport sequence—or in courses where the instructor chooses to emphasize unsteady momentum, mass, and energy transfer—a more thorough treatment of Laplace Transforms (theory of residues) can prepare the student to deal easily with more challenging problems than those belonging to the classes of examples discussed here.

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