A METHOD FOR DETERMINING SELF-SIMILARITY

Transient Heat Transfer with Constant Flux

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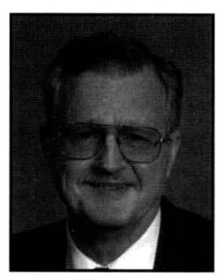
When similarity solutions to partial differential equations are introduced in the classroom, the introduction of similarity variables and the approach to self-similar problems often appears to be something of a "dark art." This paper provides an example to show how proper dimensional analysis can be used to demonstrate the existence of self-similar behavior. The procedure is as follows:

1. State the governing equations and boundary conditions.
2. Rearrange any variables that can be combined in additive or multiplicative groups to simplify the governing equations and boundary conditions. Simplifications that should always be performed if possible include: rearranging the governing equations so one term has no coefficients, translating the independent variables so inner boundary conditions are at zero, setting all but one of the boundary and initial conditions to zero by translation of the dependent variable, and cross-multiplying the boundary conditions so that they equal dimensionless constants (*e.g.*, 0, ±1, or ±∞).*[1]
3. Write the dimensional-variable space of the problem as a system of inequalities. Include dimensional independent variables, dependent variables, and system parameters that remain after performing step two. State all lower and upper bounds of these quantities. This bookkeeping measure concisely summarizes all variables and their possible values. In addition, it clearly shows variables that can be removed, or bounds that can be relaxed, during asymptotic analysis.
4. Compose a dimensional matrix for the dimensional-variable space. Determine the rank of this matrix. Subtract the rank from the total number of variable groups. If dimensionless groups arose during rearrangement in step two, add one for each. The result is the number of dimensionless degrees of freedom involved in the problem.
5. If the number of dimensionless degrees of freedom is two, a similarity solution exists. If the degrees of freedom can be reduced to two by taking upper (or lower) bounds of the independent variables to



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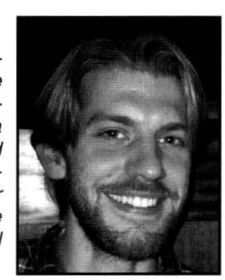

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* *In linear Dirichlet problems, the dependent variable should sometimes be made dimensionless at this point in the procedure. An example is the solution to the Navier-Stokes equation for impulsive motion of a flat plate in a semi-infinite medium (also known as Stokes' first problem, posed in Reference 1).*

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infinity (or negative infinity), a similarity solution describes this asymptotic regime.

We illustrate these steps below with the classic problem of transient constant-flux heat transfer to a stagnant one-dimensional medium between a conductive inner wall and an insulated outer wall.

The earliest experiment under the conditions analyzed here is credited to F. E. Neumann, who performed experiments to measure the thermal conductivity of solids. In 1862 he lectured in Paris, proposing mathematics to describe bars heated electrically at one end.[2] He used the heat equation (with a superfluous generation term) to obtain an expression for thermal conductivity under conditions of constant flux; for cubic bodies of low conductivity, he derived another expression to show that temperature rises with the square root of time. Preston's *Theory of Heat* references similar experiments by O. J. Lodge (1879), and gives another incorrect mathematical treatment.[3,4] The finite problem was developed accurately by Carslaw,[5] and several avenues for solution of finite and semi-infinite cases were proposed by Carslaw and Jaeger,[6] who were the first of these authors to mention a possible solution by integrated error function complement. The similarity solution was introduced as an exercise in the textbook by Bird, Stewart, and Lightfoot.[7]

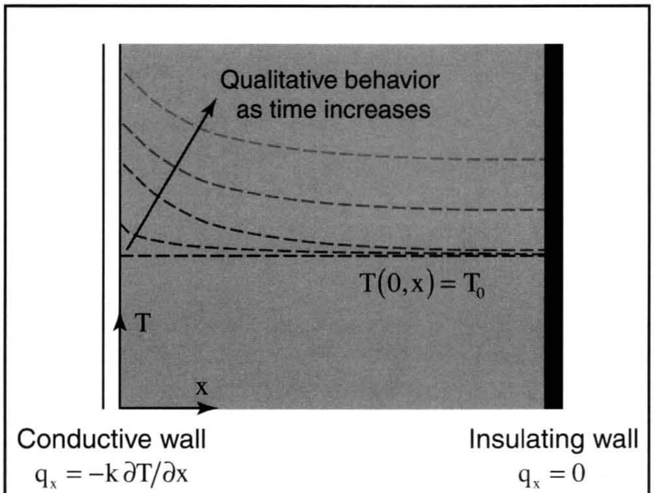

Figure 1. *Experimental geometry for the heat-transfer problem.*

Figure 1 shows a one-dimensional rectilinear region with spatially uniform initial temperature T_0 and walls at x = 0 and L. At time t = 0, a uniform and constant heat flux q_x (which may be positive or negative) is applied in the positive x-direction at the conductive boundary x = 0; the boundary at x = L is well-insulated.* We assume experimental conditions with adiabatic walls parallel to the heat flux, effectively constant and isotropic transport properties, and no homogeneous heat generation.

Three solutions, valid at long times (Eq. 18), intermediate times (Eq. 19), and short times (Eq. 20) are presented here. Dimensional considerations are then used to realize a fourth self-similar solution (Eq. 29), which describes asymptotic behavior in a semi-infinite medium or a medium observed at very short times.

STATEMENT OF GOVERNING EQUATIONS: INITIAL AND BOUNDARY CONDITIONS

We begin by writing the governing equations and boundary conditions. The transient one-dimensional rectilinear heat equation applies in this case

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where ρ is density of the medium, $\hat{C}_p$ is its specific heat capacity at constant pressure, and k is its thermal conductivity. Appropriate initial and boundary conditions are

$$T(0, x) = T_0 \tag{2}$$

$$-k \left.\frac{\partial T}{\partial x}\right|_{(t \geq 0, 0)} = q_x \tag{3}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(t, L)} = 0 \tag{4}$$

We seek mathematical solutions to Eq. (1) satisfying conditions 2 through 4 that are easily calculated at all experimental time scales.

As a first approach to simplification, we apply the second

* *To imagine a more concrete experiment, think of the wall at x = 0 as a metal block, which has high thermal conductivity, and the wall at x = L as a piece of low-density foam, both of which are impermeable to and insoluble in the thermally conductive medium between. Assume the medium is water, which is isotropic, has low viscosity, and is of intermediate conductivity. An electric heater supplies constant power to the metal. The system can be oriented with respect to gravity to suppress the effect of free convection.*

step of our procedure, which results in this restatement:

$$\frac{\partial(T-T_0)}{\partial t}=\frac{k}{\rho\hat{C}_p}\frac{\partial^2(T-T_0)}{\partial x^2} \tag{5}$$

$$T(0,x)-T_0=0 \tag{6}$$

$$\frac{k}{q_x}\left.\frac{\partial(T-T_0)}{\partial x}\right|_{(t\geq 0,0)}=-1 \tag{7}$$

$$\left.\frac{\partial(T-T_0)}{\partial x}\right|_{(t,L)}=0 \tag{8}$$

The initial condition is now zero, and the governing equation and boundary condition 3 have been rearranged. It is apparent here that T_0 appears only in an additive combination with T, and that $\hat{C}_p$ and q_x occur only in multiplicative combination with k.

STEADY STATE FOURIER-SERIES SOLUTION: LARGE-S LAPLACE-TRANSFORM SOLUTION

We now implement step three of the procedure. The *dimensional-variable space* of a problem summarizes the domains of remaining dimensional independent variables, the ranges of dimensional dependent variables and system parameters, and all known bounds of these quantities. While not essential, this step is a useful tool to help clarify one's thinking before approaching the differential equation. The dimensional-variable space of the problem stated in Eqs. (5) through (8) is

independent variables $\begin{cases}0\leq t<\infty\\0\leq x\leq L\end{cases}$

dependent variable $\{-\infty\leq T(t,x)-T_0<\infty$ (9)

parameters $\begin{cases}0\leq \dfrac{k}{\rho\hat{C}_p}<\infty\\-\infty<\dfrac{k}{q_x}<\infty\end{cases}$

Inequalities 9 reflect that physical values of the properties $\rho\hat{C}_p$ and k are positive. The flux q_x and temperature difference $T - T_0$ may take positive or negative values, because heat can be added to or taken from the system, resulting in an increase or decrease of temperature. The distance L between walls has been included as the upper bound of x.

Step four is to apply the "Buckingham pi theorem" (the rigorous development of which may be more appropriately attributed to Bridgman, and the linear algebraic formulation of which owes to Langhaar) to these groups of variables.[8,9,10,11] The dimensional matrix is

$$\begin{array}{c} \\ s\\ m\\ K\end{array}\begin{array}{c}\begin{array}{cccccc} t & x & T-T_0 & k/\rho\hat{C}_p & k/q_x & L\end{array}\\ \begin{bmatrix}1 & 0 & 0 & -1 & 0 & 0\\0 & 1 & 0 & 2 & 1 & 1\\0 & 0 & 1 & 0 & -1 & 0\end{bmatrix}\end{array} \tag{10}$$

Matrix 10 is created by putting relevant fundamental SI units to the left of the rows and elements of the dimensional-variable space above the columns. The powers to which units are raised in each variable determine the values of the matrix elements.

There are 6 groups of variables, and the rank of the dimensional matrix is 3; therefore, by the pi theorem, the problem can be phrased in a dimensionless-variable space with three degrees of freedom. If a two-dimensional boundary-value problem with three dimensionless degrees of freedom is separable and has a closed domain in one independent variable, it can usually be reduced to a Sturm-Liouville system in the closed domain if asymptotic behavior is subtracted from the initial condition. Although our goal here is to illustrate self-similarity, the Fourier-series solution and a Laplace-trans-

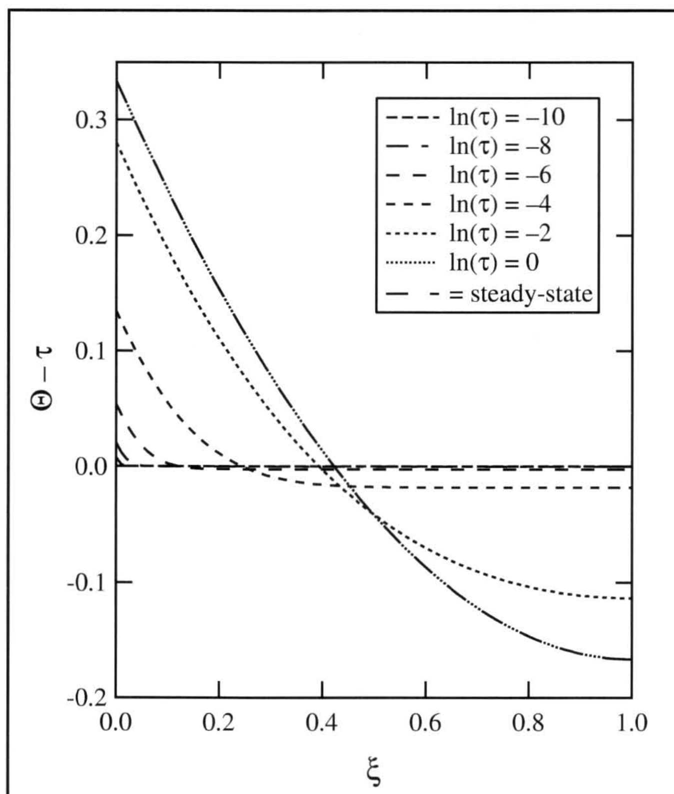

Figure 2. *Plot of the long-time solution given by Eq. 18 and the transient Fourier-series solution given by Eq. 19.*

form solution more useful at short times are shown now. The Fourier series results from the standard approach to separable partial differential equations; the next section will reveal that the Laplace-transform solution relates fundamentally to the result by similarity transformation.

At this point, the three dimensionless variables can be selected by trial and error, with two dimensionless degrees of freedom allotted to the independent variables and one to the dependent variable. A more physically sound route to a natural set of dimensionless variables is provided by an overall energy balance around the slab,

$$-t\int_S \bar{q}\cdot d\bar{S} = \int_V \rho\hat{C}_p(T - T_0)dV \qquad (11)$$

Upon simplification of the integrals, multiplication of both sides by k, and some rearrangement, this energy balance reduces to the simple form

$$\tau = \int_0^1 \Theta d\xi \qquad (12)$$

In Eq. (12)

$$\tau = \frac{kt}{\rho\hat{C}_p L^2} \qquad \xi = \frac{x}{L} \qquad \Theta = \frac{k(T - T_o)}{q_x L} \qquad (13)$$

which assigns the appropriate number of degrees of freedom to the independent variables and the dependent variable. Substituting these variables into the governing equation and boundary conditions, we find

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\xi^2} \qquad (14)$$

$$\Theta(0,\xi) = 0 \qquad (15)$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{(\tau\geq 0,0)} = -1 \qquad (16)$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{(\tau,1)} = 0 \qquad (17)$$

Note that Θ is always positive because the heat flux no longer appears as a parameter.

A first step in the analysis of a transient partial differential equation is to obtain a solution valid at long times.* Usually, long-time solutions are obtained by discarding the terms containing time derivatives, but because this problem involves constant flux of heat, and therefore a constant increase or decrease in system energy, the time derivative of the dimensionless temperature approaches a nonzero value at long times. To obtain long-time behavior when a system accumulates or loses energy, the condition

$$\left.\frac{\partial\Theta}{\partial\tau}\right|_{\tau\to\infty} = f(\xi)$$

should be employed. The solution that satisfies conditions 16 and 17 when $t\to\infty$ is then

$$\Theta_\infty(\tau,\xi) = \tau + \frac{1}{2}\xi^2 - \xi + \frac{1}{3} \qquad (18)$$

where the factor of 1/3 is included so that Θ_∞ satisfies the dimensionless energy balance given in Eq. (12). The Fourier-series solution valid at all times is

$$\Theta(\tau,\xi) = \tau + \frac{1}{2}\xi^2 - \xi + \frac{1}{3} - \frac{2}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\exp(-j^2\pi^2\tau)\cos(j\pi\xi) \qquad (19)$$

Equations (18) and (19) are plotted in Figure 2.

The rate of convergence of the Fourier series in Eq. (19) slows as $\tau\to 0$. A series that converges much more rapidly is obtained as follows. Take the Laplace transform of the problem with respect to time. A large-s expansion of this result can be obtained by Maclaurin expansion of the transformed problem with respect to $\exp(-\sqrt{s})$. Term-by-term inversion of this series by comparison with a table of Laplace transforms[12] gives an alternative to Eq. (19)

$$\Theta(\tau,\xi) = 2\sqrt{\tau}\sum_{j=0}^{\infty}\left[\text{ierfc}\left(\frac{2j+\xi}{2\sqrt{\tau}}\right) + \text{ierfc}\left(\frac{2j+2-\xi}{2\sqrt{\tau}}\right)\right] \qquad (20)$$

which converges rapidly at small values of τ and is plotted in Figure 3 (next page). The integrated error function complements included in Eq. (20) are defined as the functions which solve the differential equation

$$\frac{d^2y}{dz^2} + 2z\frac{dy}{dz} - 2ny = 0 \qquad n = -1, 0, 1, 2, \ldots \qquad (21a)$$

when n is equal to unity.

Ordinary differential Eq. (21a) is satisfied by functions of the form[13]

$$y = Ai^n\text{erfc}(z) + Bi^n\text{erfc}(-z)$$

where

* *Taking the long-time form of a transient equation to obtain an ordinary equation exemplifies a basic type of astymptotic analysis: an independent variable (t) can be removed from the variable space by assuming it takes a large value ($t\to\infty$). The governing equations and boundary conditions must then be rephrased to reflect insensitivity to this variable (accumulation becomes a function of x only). We applied this type of asymptotic simplification implicitly when reducing the problem to one spatial dimension.*

$$i^{-1}\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\exp\left(-z^2\right)$$

$$i^{0}\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty}\exp\left(-z^2\right)dz = \mathrm{erfc}(z)$$

$$i^{n}\mathrm{erfc}(z) = -\frac{z}{n}i^{n-1}\mathrm{erfc}(z) + \frac{1}{2n}i^{n-2}\mathrm{erfc}(z) \qquad (21b)$$

Solutions to Eqs. (14) through (17) given by Eqs. (19) and (20) are *identical*. Fewer terms of Eq. (19) are required for accuracy at long times, and fewer of Eq. (20) are needed at short times.

SELF-SIMILARITY IN AN ASYMPTOTIC REGIME

Previously, we used L to scale position x and time t. Step 5 of the procedure outlined in the first section of this paper yields an asymptotic result for small τ. Physically, the condition that $\tau << 1$ corresponds to systems where the length scale or volumetric heat capacity is large, or the thermal conductivity or time is small; the dimensionless energy balance given by Eq. (12) further shows that when τ is small, the dimensionless energy put into the system is also.

Under any circumstances where $\tau << 1$, L may be considered to approach infinity, the domain of x becomes open, the number of columns in the dimensional matrix reduces by one, and the degrees of freedom reduce to two. Parabolic problems that afford two dimensionless degrees of freedom can be solved by grouping the independent variables together in a single similarity variable. This condition is called *complete similarity*, or *self-similarity of the first kind.*[14]

We choose two dimensionless variables, making sure both independent variables are contained in one of them and the dependent variable is in the other:

$$\eta = \beta_1 x\sqrt{\frac{\rho C_p}{kt}} \qquad \theta = \beta_2\frac{T - T_0}{q_x}\sqrt{\frac{k\rho C_p}{t}} \qquad (22)$$

Here, the similarity variable η and dependent variable θ have been chosen because they are relatively simple forms. To put x in the numerator of η simplifies back-substitution, because second derivatives of η with respect to position are then zero. There is only one derivative with respect to time in the governing equations and boundary conditions and there are two with respect to x, which suggests choosing a θ that omits x, if possible. It should be noted that an ordinary differential equation will result for *any* choice of dimensionless similarity variable, as long as it excludes the dependent variable and contains both independent variables.

We introduced constants β_1 and β_2 into relations (22); particular values for them can be selected later to simplify solution of the resultant ordinary differential equation and put results in a standard form.

Taking $L \to \infty$ in Eqs. (5) through (8) and then inserting relations (22) give

$$0 = \frac{d^2\theta}{d\eta^2} + \frac{1}{2\beta_1^2}\eta\frac{d\theta}{d\eta} - \frac{1}{2\beta_1^2}\theta \qquad (23)$$

$$\theta(\infty) = 0 \qquad (24)$$

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = -\frac{1}{\beta_1\beta_2} \qquad (25)$$

$$\left.\frac{d\theta}{d\eta}\right|_{\eta\to\infty} = 0 \qquad (26)$$

Boundary condition (26) limits the asymptotic behavior of the solution at large x, and is not as strict as Eq. (8), which restricts the solution at a particular x. Substitution of η into Eqs. (5) through (8) as L approaches infinity to yield Eqs. (23) to (26) represents a *similarity transformation*.

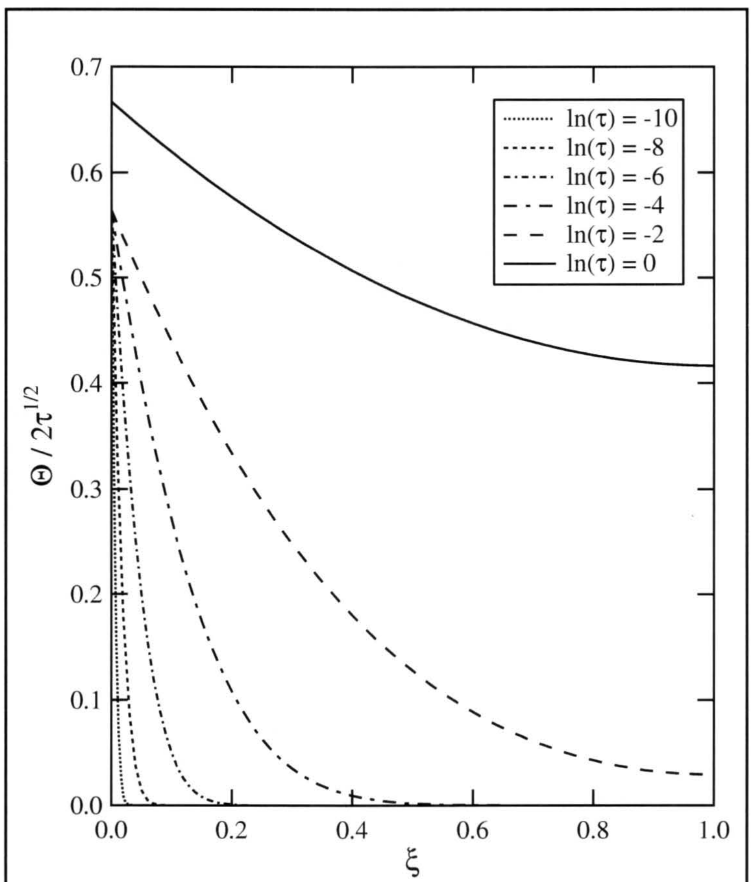

Figure 3. *Graph of the integrated error-function-complement series solution, Eq. (20).*

Note that introduction of the similarity variable has reduced governing Eq. (5) to an equation of second order. In problems that are amenable to similarity transformation, two of the boundary conditions should collapse to a single condition. Either boundary condition (24) or (26) can be discarded on the basis that it is superfluous—a solution that satisfies governing Eq. (23) and one of Eqs. (24) or (26) must satisfy the other.

If β_1 is chosen to be 1/2, then Eq. (23) matches Eq. (21a) with n = 1. Boundary condition (25) takes its simplest form when

$$\beta_1\beta_2 = 1 \tag{27}$$

To satisfy Eq. (27), we choose $\beta_2 = 2$. The dimensionless similarity variables are

$$\eta = \frac{x}{2}\sqrt{\frac{\rho C_p}{kt}} \quad \text{and} \quad \theta(\eta) = \frac{2(T - T_0)}{q_x}\sqrt{\frac{k\rho C_p}{t}} \tag{28}$$

A solution to Eqs. (5) through (8) when $L \to \infty$ is given by

$$\theta(\eta) = \operatorname{ierfc}(\eta) \tag{29}$$

Equation 29 is plotted in Figure 4. Because

$$\theta = \frac{\Theta}{2\sqrt{\tau}}$$

this solution matches the first term of series 20 when reflections from the far wall are neglected. As an exercise, the student can take the limit of series (20) when dimensionless time approaches zero to retrieve the similarity solution.

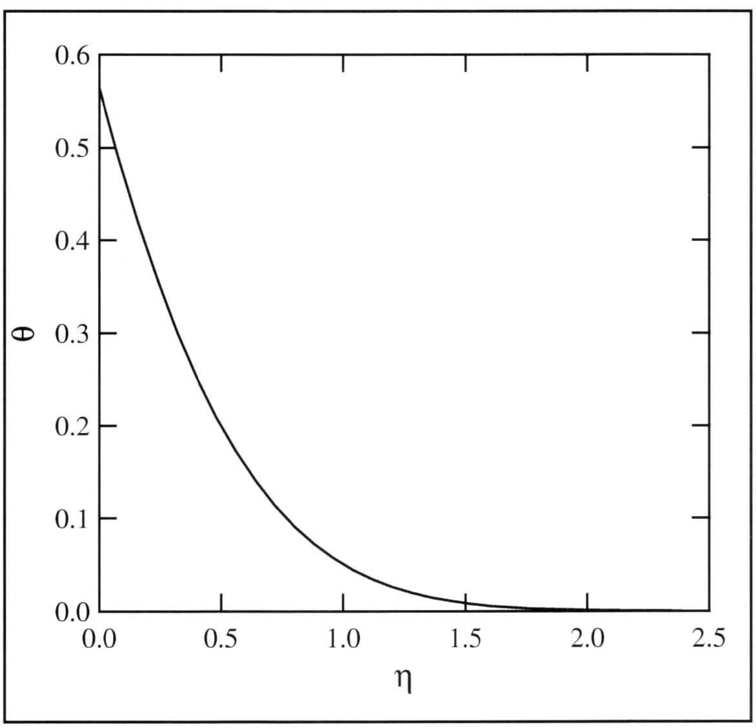

Figure 4. *The similarity solution yielded by Eq. (29).*

CONCLUSION

A methodology has been proposed that allows stepwise determination of self-similar solutions of the first kind by dimensional and asymptotic analysis. The five-step procedure is given in section 1, and is illustrated by the problem of transient constant-flux heat transfer to a stagnant medium with an insulated far wall in the remaining sections. Our approach illustrates how simplifying governing equations and boundary conditions according to certain rules and writing a dimensional matrix at the outset of a problem can effectively guide its solution.

A procedure to obtain self-similar solutions of the second kind, where the similarity variable can be used but more than two dimensionless degrees of freedom are present, will be addressed in future work. An example of a self-similar problem of the second kind is the transient mass transfer of a solute from a sphere at constant concentration into a stagnant medium in which the solute is homogeneously consumed with first-order kinetics.

ACKNOWLEDGMENTS

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