A LABORATORY EXPERIMENT ON HOW TO CREATE DIMENSIONLESS CORRELATIONS

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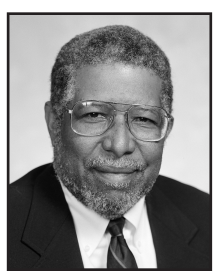
Dimensionless correlations are a prominent feature in the practice of chemical engineering. It is difficult, however, to create a student experiment that will illustrate how they are created. For instance, a pipe flow experiment can be set up where pressure drop is measured as a function of the flow rate. The results can be consistent with the published dimensionless correlation, but the results cannot be called a verification of the correlation unless all the relevant parameters have been varied.

This paper discusses a laboratory experiment in our junior-year laboratory that shows how dimensionless correlations should be constructed. Balls of various densities and diameters are dropped from various heights into a pool of water, and the maximum depth reached by the ball is recorded for each drop. The variables are the liquid density, the ball density, the ball diameter, the initial height above the liquid, and finally, the greatest depth of penetration. The experimental apparatus is shown in Figure 1.

For many years, this experiment was a great frustration to the students. They kept futilely attempting to use the Buckingham Pi Theorem to find the appropriate dimensionless groups. In fact, the Buckingham Pi Theorem is an existence theorem. It tells us that given m quantities describing a physical situation and n fundamental units (mass, length, etc.), a dimensionless description of the situation can be written as a function of m-n dimensionless groups. The proof is a construction proof wherein an algorithm is constructed to compute example m-n dimensionless groups. Students are sometimes aware that the version of the theorem found in chemical engineering texts also states that the product of any dimensionless group to any power times any other dimensionless group to any power is also a legitimate dimensionless group. What they don't always understand is that the latter creates an infinite combinatorial problem. The appropriate groups can be obtained only by a low-probability accident. The latter method will fail in situations like time-dependent heat transfer problems where the theorem predicts too many dimensionless parameters. There are more elaborate versions of the theorem in the literature that claim to guide the user in which variables and parameters can be combined, but they contain no physics and are thereby suspect as a guide to constructing a physical theory.



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1 *Prof. Robert Edwards passed away after submitting this paper. Final revisions to the paper were made by Prof. Daniel Lacks, Department of Chemical Engineering, Case Western Reserve, daniel.lacks@case.edu. Correspondence regarding this paper can be directed to Prof. Lacks.*

THEORY

The appropriate method is the one suggested decades ago by Bird, Stewart, and Lightfoot,[1] and reinforced by Sides.[2] Basically it says that the physics that control the process must be known in order to find the appropriate dimensionless groups. The method consists of writing down the governing differential equation and then making that equation dimensionless using the boundary and initial conditions. The functional forms of the governing dimensionless groups will reveal themselves after some manipulation. Importantly, this procedure is valid for approximate models where only the dominant variables are treated.

The core of this problem consists of determining the velocity of the ball in the water as a function of the ball's physical parameters, the parameters of the water, and time. When the velocity goes to zero, the ball is as deep as it is going to get (recall, the balls have a specific gravity less than 1, so they ultimately float).

It is easy to estimate the initial ball velocity in the water, v_o, when dropped from a height L, by assuming that the initial potential energy is completely converted to kinetic energy.

$$v_o = \sqrt{2gL} \tag{1}$$

where g is the gravitational acceleration. This assumes that that velocity is well under the terminal velocity and the effect of air friction is negligible. The ball will lose some speed when it penetrates the water surface, but it will be assumed that loss is negligible; recall that the purpose of this analysis is to develop a functional form to be used in a fitting procedure, rather than to solve the full equations rigorously, and as pointed out above, reasonable approximations do not damage our ability to obtain a reasonable correlation. Once in the water, the differential equation that describes the acceleration of the ball is given by Newton's second law,

$$\left(\rho_b \frac{\pi D^3}{6}\right)\frac{dv}{dt} = F_b + F_f \tag{2}$$

where v, D, and ρ_b are the velocity, diameter, and density of the ball. The buoyant force, F_b, according to Archimedes, is simply

$$F_b = \frac{\pi D^3}{6}\left(\rho_b - \rho_w\right)g \tag{3}$$

where ρ_w is the density of water. The friction force is estimated in typical chemical engineering fashion using another dimensionless correlation (see Bird, Stewart, and Lightfoot),

$$F_f = -\frac{1}{2}\rho_w v^2 \frac{\pi D^2}{4} f\left(Re\right) \tag{4}$$

where the dimensionless friction factor f is a function of the Reynolds number $Re = \left(\rho_w vD\right)/\mu$, and μ is the fluid viscosity. The Reynolds number is a function of the velocity of the ball, and it changes as the ball moves through the fluid.

The equation of motion can now be written

$$\rho_b \frac{\pi D^3}{6}\frac{dv}{dt} = \frac{\pi D^3}{6}\left(\rho_b - \rho_w\right)g - \frac{1}{8}\rho_w \pi D^2 v^2 f\left(Re\right) \tag{5}$$

In principle, this equation could be solved numerically, but it is not necessary to do so to get the appropriate dimensionless groups. By dividing Eq. (5), by the parameters on the left-hand side and g, and defining the dimensionless velocity $v^* = v/\sqrt{2gL}$, the result is

$$\sqrt{\frac{2L}{g}}\frac{dv^*}{dt} = \frac{\left(\rho_b - \rho_w\right)}{\rho_b} - \frac{3}{2}\frac{\rho_w}{\rho_b}\frac{L}{D}v^{*2} f\left(Re_0 v^*\right) \tag{6}$$

where $Re_0 = \left(\rho_w \sqrt{2gL}D\right)/\mu$ is the initial Reynolds number as the ball enters the fluid. Since all the terms on the right-hand side are already dimensionless, the left-hand side must be as well. By defining dimensionless time as $t^* = t\sqrt{\frac{g}{2L}}$, the equation obtained is

$$\frac{dv^*}{dt^*} = \left(1 - \frac{\rho_w}{\rho_b}\right) - \frac{3}{2}\frac{\rho_w}{\rho_b}\frac{L}{D}v^{*2} f\left(Re\right) \tag{7}$$

Eq. (7) implies that the solution for the dimensionless velocity will be of the form

$$v^* = v^*\left(t^*; \frac{\rho_w}{\rho_b}, \frac{L}{D} Re_0\right) \tag{8}$$

which demonstrates the dimensionless groups that affect the trajectory of the ball.

The maximum depth the ball reaches, h, is obtained as

$$h = \int_0^{t_{max}} v\left(t\right)dt \tag{9}$$

where t_{max} is the time at which the ball reaches its maximum depth. Using the definitions for the dimensionless velocity and time, this leads to

$$h = \sqrt{2gL}\sqrt{\frac{2L}{g}}\int_0^{t^*_{max}} v^*\left(t^*\right)dt^* \tag{10}$$

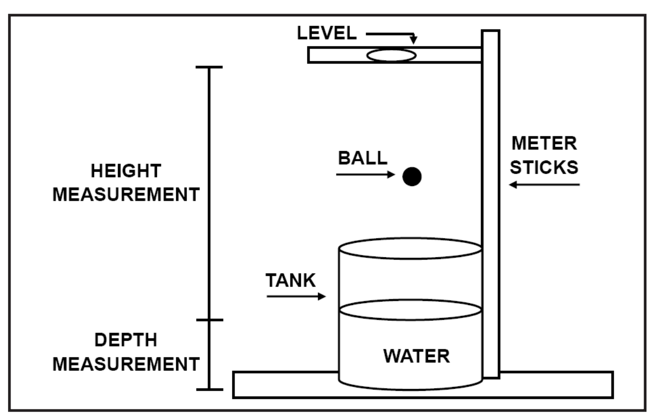

Figure 1. *Ball-Dropping Apparatus.*

and further simplification leads to

$$\frac{h}{L} = 2\int_0^{t^*_{max}} v^*\left(t^*\right)dt^* \qquad (11)$$

Eq. (11) implies that

$$\frac{h}{L} = \frac{h}{L}\left(t^*_{max}; \frac{\rho_b}{\rho_w}, \frac{L}{D}, Re_0\right) \qquad (12)$$

Now, the value of t^*_{max} is determined by the condition $v^*(t^*_{max})=0$, which in combination with Eq. (8) implies

$$t^*_{max} = t^*_{max}\left(\frac{\rho_b}{\rho_w}, \frac{L}{D}, Re_0\right) \qquad (13)$$

By combining the above two equations, the result is found that

$$\frac{h}{L} = \frac{h}{L}\left(\frac{\rho_b}{\rho_w}, \frac{L}{D}, Re_0\right) \qquad (14)$$

which shows how the dimensionless depth to which the ball falls will depend on a set of dimensionless physical parameters.

At this point, the solution is not known, but common engineering practice is to attempt a *correlation* of the form

$$\frac{h}{L} = A\left(\frac{\rho_b}{\rho_w}\right)^a \left(\frac{L}{D}\right)^b \left(Re_0\right)^c \qquad (15)$$

where the constants A,a,b,c are the parameters that minimize a least squares sum. There is not enough space here to cover the details, but the fit should never be done by taking the logarithm of the expression above and doing a least squares fit to the linearized expression; the result would be biased coefficients. Modern computer power makes the computation simple using an optimizer like "Solver" in Excel®.

One final question. Having used the computer to estimate the fit parameters, how robust are those estimates? In principle, if the experiment is repeated, different parameters will result. The important question is "How large a variation is expected from experiment to experiment?" The answer to this question also affects the appropriate number of significant figures it makes sense to report. The method used here is the maximum likelihood estimate[3] of the parameter variance for the case where the errors in the depth estimate are Gaussian. Under these circumstances, the variance of the parameters from experiment to experiment is proportional to the average measurement error and inversely proportional to the sharpness of the least squares minimum, taken as the expected second derivative at the minimum. It can be shown that the parameter estimate error can be estimated by the following procedure[3]: First, the Fisher information matrix,

$$F_{pq} = \sum_n \frac{1}{s^2} \left.\frac{\partial y}{\partial \alpha_p}\right|_n \left.\frac{\partial y}{\partial \alpha_q}\right|_n \qquad (16)$$

is estimated, where y is the function being fit [Eq. (15)], the α_p are the parameters of the fit [A, a, b, c in Eq. (15),] and the derivatives are evaluated at the set of parameters (Re_0, D, and ρ_B) used for the n^{th} experimental trial. The parameter s^2 is the estimated measurement variance, which is the average square of the deviation of the theory from the measurement estimated after the successful fit. Then, the estimate of the reproducibility variance for each of the parameters is given by

$$\sigma_p^2 = F_{pp}^{-1} \qquad (17)$$

where F_{pq}^{-1} is the inverse matrix of F_{pq}. This methodology is described in detail elsewhere.[3] While these calculations look cumbersome, they are easily done using a spreadsheet program like Excel®.

RESULTS AND DISCUSSION

A randomly selected student's set of laboratory data is analyzed as described above. This data included the maximum depths reached by 14 different balls (with various D and ρ_B) dropped into water from 3 separate heights above the water; the parameter ranges were 2.5 cm < D < 6.5 cm, 0.55 g/cm^3 < ρ_B < 0.90 g/cm^3, and 34 cm < L < 141 cm, and the maximum depths that the balls reached were in the range 11 cm < h < 47 cm. This data was fit with Eq. (15), and Figure 2 shows a plot of the measured value of the depth vs. the theoretical depth obtained from the fit above. As is apparent from the graph, the fit is an overall success. To quantify the uncertainties, the Fisher information matrix and its inverse are calculated,

$$F_{pq} = \begin{pmatrix} 704.93 & -331.06 & 2973.0 & 13571 \\ -331.06 & 198.01 & -1437.0 & -6388.8 \\ 2973.0 & -1437.0 & 1311.0 & 57099 \\ 13571 & -6388.8 & 57099 & 262630 \end{pmatrix}$$

$$F_{pq}^{-1} = \begin{pmatrix} 0.3399 & -0.00542 & -0.01138 & -0.01522 \\ -0.00542 & 0.02554 & 0.00194 & 0.00048 \\ -0.01138 & 0.001937 & 0.00194 & 0.00021 \\ -0.01522 & 0.00048 & 0.00021 & 0.00076 \end{pmatrix}$$

to obtain a final result for the fit of the dimensionless correlation to the experimental data,

A= 1.60 ± 0.51

a= 1.32 ± 0.14

b= -0.61 ± 0.04

c= 0.062 ± 0.024

Note that the uncertainty estimates listed are the σ_p, which describe the standard deviations for the uncertainties in the parameters, such that ranges of $\pm 1\sigma_p$ are associated with 68% confidence limits, and $\pm 2\sigma_p$ are associated with 95% confidence limits, etc.

Students often try other functional forms, however, which are not dimensionless, for the fitting. For instance, from this same data set, the student reports that the fit

$$h = 0.3089\text{cm}\left(D^{0.79}\right)\left(\rho_b^{1.06}\right)\left(L^{0.26}\right) \tag{18}$$

is even better than the fit to Eq. (15), in that the R^2 calculated with this fit is larger. Students typically do not know what R^2 is a measure of, much less how large a change is meaningful. That should be pointed out to students, but even more important is that they should recognize that this second fit would tell them nothing about what might happen if the experiment were carried out using a Newtonian oil rather than water. The first correlation should do a good job of predicting the change in behavior since it is dimensionless and contains the essential physics of the problem, including relative densities and the effect of viscosity.

The method demonstrated to derive the appropriate dimensionless groups above is quite robust, but it can give different functional forms. For instance, if the velocity were made dimensionless by embedding it in the Reynolds number, a different form would have been obtained. Following the same methodology as above, the form for the correlation would be

$$\frac{\rho_w^2 hgD^2}{\mu^2} = A'\left(\frac{\rho_b}{\rho_w}\right)^{a'}\left(\frac{\mu^2}{\rho_w^2 gD^3}\right)^{b'}\left(\frac{\rho_w\left[\sqrt{2gL}\right]D}{\mu}\right)^{c'} \tag{19}$$

With no loss of generality, this expression can be multiplied by the second dimensionless group to give

$$\frac{h}{D} = A'\left(\frac{\rho_b}{\rho_w}\right)^{a'}\left(\frac{\mu^2}{\rho_w^2 gD^3}\right)^{b'}\left(\frac{\rho_w\left[\sqrt{2gL}\right]D}{\mu}\right)^{c'} \tag{20}$$

This functional form is legitimate, but not as desirable as Eq. (15) because it stresses the viscosity, which was not varied in the experiment. In contrast, in Eq. (15), the viscosity only appeared in the friction factor correlation where the dependence on the Reynolds number is well tested.

All three versions of the correlation have the same dependence, within experimental error, on the variables used: the ball density, ball diameter, and height of the drop. The dimensionless versions are preferred if only because they give the engineer some guidance as to what experiments need to be done to firm up the correlation and what may happen when liquids other than water are used. More experiments should be done where the surface tension and viscosity are varied.

The example shown here gives an indication what information can be extracted from experimental data. Another instructor has used this same experiment as a vehicle to demonstrate randomness in experiments and examines the sample size dependence for the uncertainty in the estimate of the mean, and whether or not the probable distribution is Gaussian.

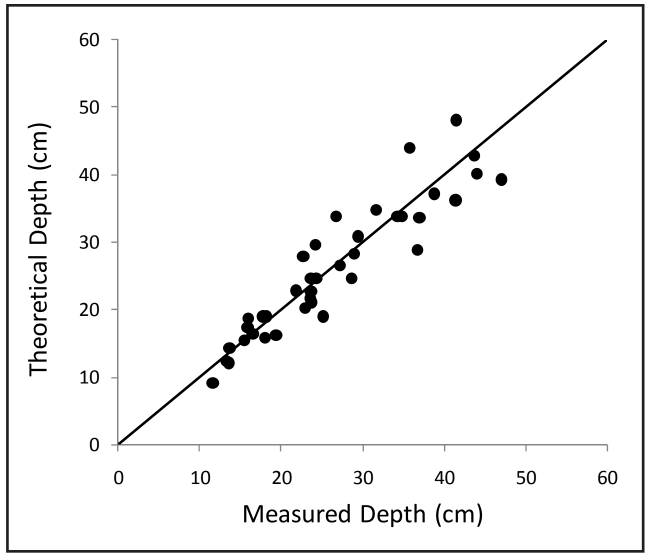

Figure 2. *Relationship between the measured maximum depth of the dropped balls and the fitted (theoretical) values. The line denotes points of equality of the measured and theoretical values.*

NOMENCLATURE

A	proportionality constant in correlation
a, b, c	exponents of dimensionless terms in correlation
D	diameter of ball
f	friction factor for a sphere
F_b	buoyancy force
F_f	frictional force
F_{ij}	ij[th] element of the Fisher information matrix
g	gravitational acceleration
h	maximum depth reached by ball
L	height ball is released above water
Re	Reynolds number
Re_0	Reynolds number at entrance to fluid
s^2	Estimated measurement variance
t	time
t*	dimensionless time $t\sqrt{g/2L}$
v	velocity of the ball
v*	dimensionless velocity of ball $v/\sqrt{2gL}$

Greek Characters

α_k	k[th] fitting parameter
ρ_b	density of ball
ρ_w	density of fluid

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