

The object of this column is to enhance our readers' collections of interesting and novel problems in chemical engineering. We request problems that can be used to motivate student learning by presenting a particular principle in a new light, can be assigned as novel home problems, are suited for a collaborative learning environment, or demonstrate a cutting-edge application or principle. Manuscripts should not exceed 14 double-spaced pages and should be accompanied by the originals of any figures or photographs. Please submit them to Dr. Daina Briedis (e-mail: briedis@egr.msu.edu), Department of Chemical Engineering and Materials Science, Michigan State University, East Lansing, MI 48824-1226.

The Break-Even Radius of Insulation **COMPUTED USING EXCEL SOLVER AND WOLFRAMALPHA**

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Most students of heat transfer will be familiar with the apparently paradoxical phenomenon whereby adding insulation to a hot cylindrical pipe (or hot wire) can increase the rate of heat loss from that pipe. Of course, with a bit of thought, it is easily seen that this is a result of the balance between reduced temperature driving force and increased heat transfer area. The heat transfer rate reaches its maximum value at the critical radius of insulation and the derivation of an expression for this radius is well-known, at least for the simplest case where the heat transfer coefficient describing convective heat transfer from the insulation to the air is constant.

Of considerably more interest, however, is the break-even radius of insulation. In Figure 1 (page 186), the relationship between heat loss per unit length and insulation radius is given.

In the plot in Figure 1, the outer pipe radius was 8 mm while the inner radius was 6.5 mm. The pipe was assumed to be made from copper with a thermal conductivity of 43 W/m.K while the thermal conductivity of the insulation material was 0.05 W/m.K, a value that is typical of a range of modern insulation materials.^[1] The heat transfer coefficient for convective heat loss was 5 W/m².K, a value that represents

the lower end of the range for natural convection heat transfer. The temperature difference between the inner pipe wall and the air was 100 Celsius degrees.

The break-even radius is the radius at which the heat loss from the insulated pipe is the same as that from the non-insulated pipe and, in Figure 1, it occurs at a radius somewhere between 12 and 13 mm. For insulation to be effective, the insulation radius must be greater than this break-even radius. Any radius less than this value will increase the rate of heat loss. In this paper we explore two ways in which the break-even radius can be calculated.

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THE BREAK-EVEN THICKNESS OF INSULATION

In any undergraduate heat transfer textbook,^[1] it will be shown that the heat loss from an insulated pipe is given by the expression

$$\frac{Q}{2\pi L} = \frac{T_1 - T_{\text{air}}}{\left(\ln \frac{R_2}{R_1}\right)/k_p + \left(\ln \frac{R_{\text{ins}}}{R_2}\right)/k_{\text{ins}} + \frac{1}{R_{\text{ins}} h}} \quad (1)$$

where Q is the rate of heat loss, L is the pipe length, R_1 is the inner pipe radius, R_2 is the outer pipe radius, R_{ins} is the outer radius of the insulation layer, k_p is the thermal conductivity of the pipe, k_{ins} is the thermal conductivity of the insulation, and h is the heat transfer coefficient describing the convective heat loss. The parameter, T_1 , is the temperature at the inner wall of the pipe and T_{air} is the air temperature.

The value of R_{ins} for which the heat transfer rate is a maximum is called the critical radius of insulation. We can work this out analytically by differentiation. The maximum rate of heat loss occurs when

$$\frac{d}{dR_{\text{ins}}} \left(\frac{T_1 - T_{\text{air}}}{\left(\ln \frac{R_2}{R_1}\right)/k_p + \left(\ln \frac{R_{\text{ins}}}{R_2}\right)/k_{\text{ins}} + \frac{1}{R_{\text{ins}} h}} \right) = 0 \quad (2)$$

Differentiating and solving (assuming constant h) gives the simple and well-known result

$$R_{\text{crit}} = \frac{k_{\text{ins}}}{h} \quad (3)$$

Now let us consider the situation where the pipe is not

insulated. In that case we have

$$\frac{Q}{2\pi L} = \frac{T_1 - T_{\text{air}}}{\left(\ln \frac{R_2}{R_1}\right)/k_p + \frac{1}{R_2 h}} \quad (4)$$

At the break-even radius of insulation, R_{BE} , the rate of heat loss from the insulated pipe must be the same as the non-insulated pipe. Therefore, Eqs. (1) and (4) can be combined to give

$$\left(\ln \frac{R_{\text{BE}}}{R_2}\right)/k_{\text{ins}} + \frac{1}{R_{\text{BE}} h} = \frac{1}{R_2 h} \quad (5)$$

This can be simplified to give

$$\ln \frac{R_{\text{BE}}}{R_2} - \frac{k_{\text{ins}}}{h R_2} \left(1 - \frac{R_2}{R_{\text{BE}}}\right) = 0 \quad (6)$$

This can also be written

$$\ln \frac{R_{\text{BE}}}{R_2} - \frac{R_{\text{crit}}}{R_2} \left(1 - \frac{R_2}{R_{\text{BE}}}\right) = 0 \quad (7)$$

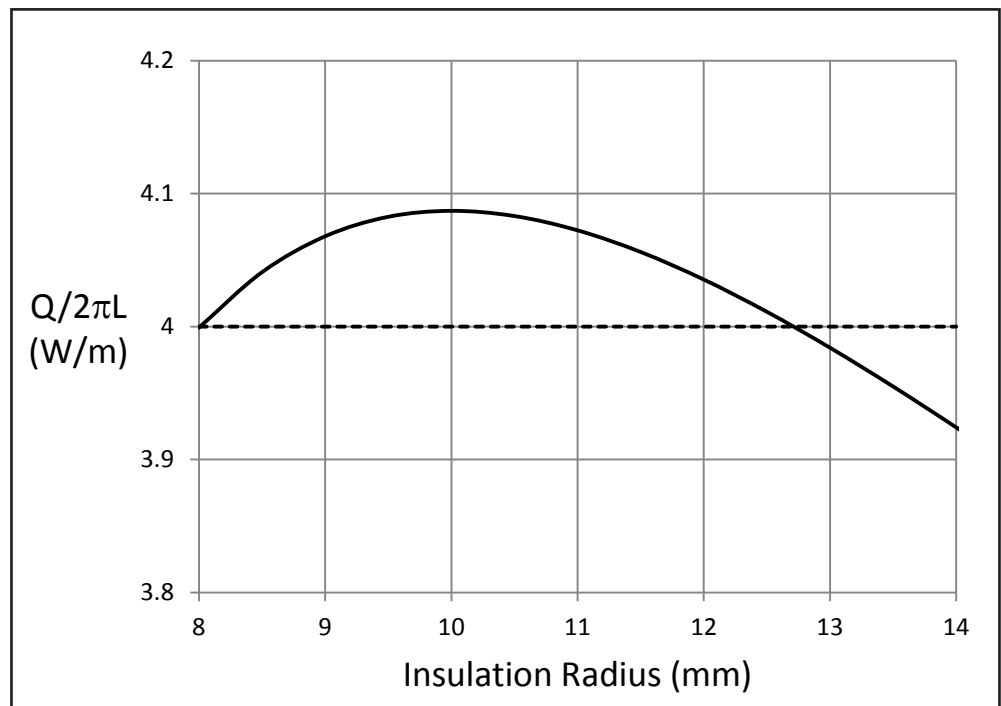
To keep things as simple as possible, we define dimensionless variables as follows:

$$x = \frac{R_{\text{BE}}}{R_2} \quad (8)$$

and

$$a = \frac{R_{\text{crit}}}{R_2} \quad (9)$$

Figure 1. Relationship between heat loss per unit length and radius of insulation.



Thus Eq. (7) becomes

$$\ln x + \frac{a}{x} - a = 0 \quad (10)$$

This is a non-linear algebraic equation that is easily solved numerically (iteratively) using any number of software tools.

NUMERICAL SOLUTION WITH EXCEL SOLVER®

Using the parameter values employed in generating Figure 1, we have

$$a = \frac{k_{ins}/h}{R_2} = \frac{0.05/5}{0.008} = 1.25 \quad (11)$$

Therefore, Eq. (10) becomes

$$\ln x + \frac{1.25}{x} - 1.25 = 0 \quad (12)$$

This can be coded into cell B1 of an Excel spreadsheet as

$$=LN(A1)+1.25/A1-1.25$$

where cell A1 contains the initial guess which in this example was given a value of 3. In Solver, we click the appropriate cells to Set Objective \$B\$1 to the Value of 0 by Varying cell \$A\$1. The solution converges to

$$x = 1.5908$$

Using Eq. (8) this implies that

$$R_{BE} = 1.5908 \times 0.008 = 0.0127 \text{ m} \quad (12a)$$

This is consistent with Figure 1. Of course the same answer can be obtained by using any number of well-known software packages including the Goal Seek tool in Excel, MATLAB®, and POLYMATH®. But, as with any numerical solution of non-linear algebraic equations, the initial estimate must be chosen with care to ensure convergence to a meaningful answer.

SOLUTION WITH THE LAMBERT W FUNCTION

One of the more transformational technologies that have been developed in recent years is computer algebra or symbolic computation. Here we show that computer algebra reveals an unexpected pseudo-analytical expression for the break-even radius of insulation. There are a number of commercial symbolic algebra systems available, including Mathematica®, the symbolic toolbox of MATLAB®, MAPLE®, and a number of other systems available free on the Internet. Here we use the on-line knowledge engine, WolframAlpha^[2] which is powered by Mathematica®. To solve Eq. (10) we simply enter the following “command” into the window on the WolframAlpha site^[2]:

$$\text{solve}(\log(x)+a/x-a==0, x)$$

The following result is returned:

$$x = -\frac{a}{W(-ae^{-a})} \quad (13)$$

In this expression, W is the Lambert W function, also known as the Omega Function or the Product Logarithm. To illustrate the origins of this function, consider the following non-linear algebraic equation

$$xe^x - k = 0 \quad (14)$$

where k is defined on the real domain $(-1/e, \infty)$. Further details on this function can be found at WolframAlpha.^[2] Then

$$x = W(k). \quad (15)$$

Evaluation of the W -function typically requires some sort of iterative algorithm so at first glance it would seem that introducing this function is somewhat pointless. However, this approach turns out to be quite useful because $W(x)$ can be computed very easily with commonly used computational software. The W -function can be computed with Mathematica®, WolframAlpha, or MATLAB® using the `productlog(x)` function, or with Maple® using the `LambertW(x)` function. In a computational sense, therefore, the Lambert W function is no different from any of the many elementary functions that are encountered routinely in chemical engineering calculations.

For the same parameters as before, the break-even radius of insulation is thus given by

$$x = -\frac{1.25}{W(-1.25e^{-1.25})} = -\frac{1.25}{-0.7858} = 1.5907 \quad (16)$$

This is essentially the same answer as obtained with Solver, the small difference at the fourth decimal place being due to round-up errors.

This simple exercise, namely computing the break-even radius of insulation, provides a nice example of how non-linear algebraic equations arise in chemical engineering. Not only does it give the students an interesting heat transfer problem with which to practice their ability to solve these equations with commercial software, it also introduces them to the world of symbolic computation in a way that is educational as well as labor-saving. A nice additional exercise would be to assign them the task of writing Eq. (10) in the form of Eq. (14).

CONCLUSIONS

Here we have shown two methods to calculate the break-even radius of insulation, a parameter that is of more practical importance than the critical radius of insulation despite the fact that the latter is routinely covered in textbooks while the former is not. Furthermore the exercise presented here shows that use of modern computer algebra systems reveals special functions with which many engineering students might be unfamiliar. But the unfamiliarity of these functions should not deter us from using them as long as they are easy to evaluate and as long as they make process calculations easier. While it can be argued that an over-reliance on computer algebra systems might contribute, in the long term, to a general de-

cline in students' basic mathematical technique, there is no doubt that such tools provide enormous scope for learning by opening students' eyes to a world of mathematics that they are currently unlikely to encounter in much of their formal engineering education. As educators, we need to use these tools in an intelligent way, not just as a means of avoiding laborious algebraic manipulations, but as a means of advancing our students' mathematical knowledge. Furthermore, online tools like WolframAlpha, that are available as smartphone

apps, provide an attractive environment for learning quite advanced topics in engineering computation in a way that is very much in tune with modern culture.

REFERENCES

1. Holman, J.P., *Heat Transfer*, 7th Ed., McGraw-Hill, London (1992)
2. Available online at <<http://www.wolframalpha.com>> [Accessed January 2014] □