Class and Home Problems (CHP) present scenarios that enhance the teaching of chemical engineering at the undergraduate or graduate level. Submissions must have clear learning objectives. CHP papers present new applications or adaptations that facilitate learning in specific ChE courses. Submit CHP papers through journals.flvc.org/cee, include CHP in the title, and specify CHP as the article type.

ALGEBRAIC SCALING: QUANTIFYING THE ART OF MAKING GOOD ASSUMPTIONS

Paul Scovazzo

INTRODUCTION

"When all else fails, assume the answer into existence."
is Paul's Second Rule of Engineering.

I wrote the "Rules of Engineering" as an undergraduate, and my second rule reflects the seemingly arbitrary nature by which professors made assumptions to simplify equations, thereby allowing their solution. For example, why is elevation change, Δz, critical for water turbine problems but ignored in steam turbine problems? Why does the problem statement not contain information to calculate the potential energy change, when potential energy is in the governing equation? My professor would say to make a "reasonable assumption." To the professor, making "reasonable assumptions" may come from years of practice. However, how do students learn the skill of making assumptions?

Investigation of methods for the simplification of mathematical models has a history measured in decades.[1] Methods used include dimensional analysis, scaling, and inspectional analysis.[1-4] Whitaker defined three levels of assumptions and placed the teaching of assumptions in the context of developing engineering skills in students.[3] According to Whitaker, my professor was giving a Level I Assumption, the precise mathematical equality needed to solve the problem; i.e. elevation change, $\Delta z = 0$. What the student wanted was a Level II or Level III Assumption, with Level II being a comparison between terms and Level III a constraint on the physical situation that the student can apply before solving the problem.[3] Level I tells a student what is mathematically being done; while Level III gives a range of validity for the problem solution.[3]

Whitaker submitted that the discussion of Level II and Level III Assumptions encourages students to develop their own assumptions, restrictions, and constraints. Papadopoulos et al. stressed the need for engineering students to develop skills in establishing and critiquing assumptions in their study of promoting holistic problem-solving pedagogy.[5] They found that students in introductory courses have difficulty formulating necessary assumptions and often use insufficient or irrelevant information.[5] They further stated that paying attention to assumptions stimulates critical thinking in addition to promoting deeper understanding and long-term retention of concepts.[5] Therefore, making assumptions to simplify problems is part of what I call "the engineering method," and teaching assumption-making skills should be part of engineering education.[2-5]

Pedagogical methods for equation simplification exist for upper- and graduate-level course material. These methods are for the simplification of models made up of differential

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equations, such as the system of equations found in classic transport phenomena textbooks.[1-4, 6] However, this author has not encountered, nor been made aware of, equation simplification methods for undergraduate (UG) course material where the governing/design equations are algebraic. The UG literature states the need to discuss the origin and importance of assumptions, even though textbooks do not routinely contain these discussions.[5] For example, a classical thermodynamic text that I have used limits discussion of assumptions to a table of 32 typical steady-state flow devices with their typical Level I Assumptions for each device.[7]

The pedagogy of teaching engineering, therefore, needs tools to illustrate what are good assumptions. What follows is one potential tool for this pedagogical task. This tool is adaptable to courses across the engineering curriculum from introductory courses through more advanced courses such as Mass/Energy Balances, Thermodynamics, Fluid Dynamics, and Mass Transfer. In fact, the author has used this method for decades in all of these courses. The course levels ranged from sophomore to graduate with students from numerous majors such as chemical, civil, electrical, environmental, geological, and mechanical engineering. This tool is ***algebraic scaling***, and it is a method to quantify the phrase "negligible." Across the engineering curricula, terms are assumed to be "negligible."

Algebraic scaling first converts an algebraic governing equation into a dimensionless equation with a number of dimensionless groups. The method then performs an order-of-magnitude analysis of the dimensionless groups. It is proper to acknowledge the inspiration of Dr. William B. Krantz's work for the material in this manuscript. Dr. Krantz and his co-author focused on the scaling of differential equation systems, including their respective boundary conditions.[2,4,8] There are significant differences in the methods needed for the scaling analysis of differential equations vs algebraic scaling. However, the one major overlap in the methods is the creation of dimensionless groups bounded between zero and "more or less 1." This *little oh of 1* [i.e., $o(1)$] does not mean that the groups are essentially 1. In $o(1)$ scaling [4] the dimensionless groups are less than 1 or within an order of magnitude of 1; for example, $3 \approx 1$. In algebraic $o(1)$ scaling, the objective is to create a dimensionless equation where the largest groups are within an order of magnitude of 1. The key logic is that if the largest group is approximately 1 then any group with a value < 0.01 can be ignored, incurring only a 1% error. In other words, "negligible."

What follows is a classroom article written in an inductive teaching module format. The learning objectives of the presented teaching module are that at the completion of the module a student will be skilled in:

- Creating dimensionless equations where all of the groups are on the order of one
- Justifying assumptions using Level III Assumption criteria (i.e., range of validity)
- Using the dimensionless scaled equations to produce a solution even when the system of equations is underdefined
- Understanding the relationship of significant figures to the precision of the calculation

Also presented are the following skills that the algebraic scaling method can teach:

- Using the dimensionless scaled equations to determine engineering design specifications
- Using the dimensionless scaled equations to determine critical system parameters

This manuscript will explain algebraic $o(1)$ scaling by taking the reader on a journey similar to the process I use to explain the method to students. The first example is of a situation for which the students would have a good physical feel. This first example introduces the basics of the method and illustrates how the method can produce useful solutions for underdefined problem statements — "How to justify good assumptions." The second example is easy enough to be solved by teams of students during a class. The third example is for a situation where the students cannot rely on their past experience. The remaining examples in the manuscript cover the breadth of an engineering curriculum from incompressible fluid flow through thermodynamics to mass transfer.

It is also proper to clarify that the methods in this manuscript cannot be used for scaling analysis of processes modeled with differential equations. Therefore, instructors of senior/graduate courses should also familiarize themselves with Dr. Kantz's work. While there are significant differences between Dr. Krantz's work and the material presented here, there is an overlap in the motivations for the two scaling methods — namely, student confusion on the proper manner to simplify exact equations and that conventional engineering curriculum give students little or no practice in making simplifying assumptions.[2,4]

INCOMPRESSIBLE FLUID FLOW: THE INTRODUCTORY EXAMPLE

Many engineering courses develop and use some form of a mechanical energy balance equation for the flow of incompressible fluids such as

$$\frac{\Delta P}{\rho}+\frac{\Delta v^2}{2}+g\Delta z+F=-\frac{W_s}{m} \quad (1)$$

where P is fluid pressure, ρ is fluid density, v is fluid velocity, g is gravitational acceleration, z is elevation relative to the gravity field, F is friction losses, m is mass flow rate, and W_s is shaft work (i.e., pumps or turbines) defined as positive for work the system does on the surroundings. Bird, Stewart, and Lightfoot [6] call Equation 1 Bernoulli's Equation; however, other authors reserve the Bernoulli nomenclature for when $F = W_s = 0$. For convenience of communications, I will call Eq. 1 the generalized Bernoulli Equation.[a]

The generalized Bernoulli Equation meets the criteria to be an initial introduction to algebraic scaling in that students generally have a good physical understanding for its variables; namely, pressure, velocity, and elevation. Therefore, in class after the introduction/development of the generalized Bernoulli Equation, use the following example problem to illustrate the basic application of Equation 1.

Problem Statement (see Figure 1): Hydroelectric power is a renewable energy source; however, it received a poor environmental reputation in the United States in the last quarter of the 20th century. The environmental issues include the destruction of natural habitat and the flooding of large land acreages. In general, hydroelectric power comes from passing large quantities of water through turbines. The power generation may be estimated from the generalized Bernoulli Equation. For instance, the Hoover Dam is 726 ft high and flooded 146,000 acres. The Hoover Dam can generate 2.8 million kW of power by passing the flow of the Colorado River through its turbines.

However, there is a proposed environmentally "benign" hydroelectric power project that would require negligible flooding of land. The following, in brief, is the proposal:[9] divert a portion of the outflow from Lake Titicaca, a large navigable lake at 12,500 ft (3810 m) above sea level, into a water tunnel drilled through the Western Crown of the Andes.[b] At the other end of this tunnel, the water would drop down to the Pacific Ocean (approximately 200 miles distance) through a series of turbines (see Figure 1). Before committing to drilling the tunnel, your boss asks you to evaluate this proposal in "order of magnitude" terms.

> Part a: How much water (cubic meters/sec) must be diverted from Lake Titicaca to achieve the same power output as the Hoover Dam? Considering that this is an "order-of-magnitude calculation," you may assume that the exit pipe diameter is very large (i.e. exit fluid velocity ≈ 0) and ignore the friction losses.

With the given information of $\rho \approx 1000$ kg/m^3, P(lake) = 0.63P(ocean), v(lake) = 0, v(exit) ≈ 0, and $F \approx 0$, the students should calculate a $m \approx 75{,}000$ kg/s or a required volumetric flow rate of 75 m^3/s. The confirmation of this result is left up to the reader.

> Part b: Now ask the students to estimate the diameter of the tunnel/pipe system's exit to judge if this construction is feasible. Emphasize to the students while they may still assume $F \approx 0$, they cannot assume v(exit) ≈ 0. Therefore, $\Delta v^2/2$ does not equal zero, meaning the assumptions that resulted in $m \approx 75{,}000$ kg/s are no longer valid.

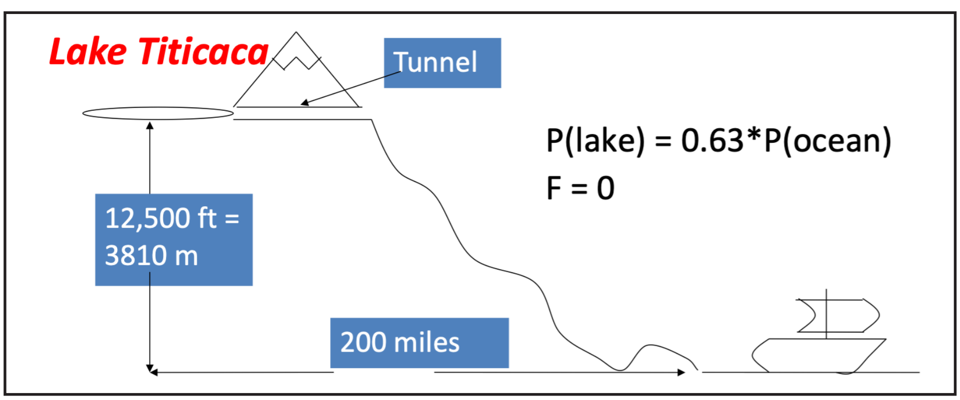

Figure 1. *Lake Titicaca hydroelectric power problem diagram.*

a. Eq. 1 does not include energy terms that most engineers routinely assume to be negligible without even acknowledging their existence, such as electric force fields or magnetic body forces (i.e. the ferrohydrodynamic Bernoulli Equation).[12]

b. This proposal has the necessary dramatic effect for this student problem statement. However, it is impractical. Geographically and politically, a more practical proposal is to divert the flow east through a series of hydroelectric stations with the last outfall being in the Bolivian portion of the Amazon basin. Also, the power generation target should be set after a hydrology study of the watershed and not by the desire to match the Hoover Dam.

After an appropriate time interval for the students to get frustrated, gather back their attention for the following lecture material.

The rewriting of Equation 1 for $F \approx 0$

$$\frac{\Delta P}{\rho} + \frac{\Delta v^2}{2} + g\Delta z = -\frac{W_s}{m} \qquad (2)$$

produces a system with one equation and two unknowns, v(exit) and m. We can write an equation for the outlet velocity in terms of mass flow rate

$$v = \frac{4m}{\rho[\pi(D^2)]} \qquad (3)$$

However, plugging Equation 3 into Equation 2 still leaves two unknowns; diameter (D) and m.

Things were easier when $\Delta v = 0$ as in Part a. How about if $\Delta v^2/2 \approx 0$? Well, then the volumetric flow rate would be the same as Part a. For an engineer, $\Delta v^2/2 \approx 0$ would mean it is very small compared to the other terms.

In scaling's simplest form, you divide through an algebraic equation by its largest term, i.e., the largest actor. The $g\Delta z$-term is the equation's largest actor. After all, we are talking about $\Delta z = 12{,}500$ ft! The result is a scaled algebraic equation made up of groups such as the following:

$$\frac{\Delta P}{\rho(g\Delta z)} + \frac{\Delta v^2}{2(g\Delta z)} + \frac{(g\Delta z)}{(g\Delta z)} = -\frac{W_s}{m(g\Delta z)} \qquad (4a)$$

$$\begin{matrix} Pressure \\ Group \end{matrix} + \begin{matrix} Kinetic \\ Group \end{matrix} + \begin{matrix} Potential \\ Group \end{matrix} = \begin{matrix} Work \\ Group \end{matrix}$$

$$\frac{\Delta P}{\rho(g\Delta z)} + \frac{\Delta v^2}{2(g\Delta z)} + 1 = -\frac{W_s}{m(g\Delta z)} \qquad (4b)$$

Two properties about these groups:

1. Each "Group" is dimensionless
2. Each "Group's" magnitude is ≤ 1

Each group on the left-hand side of Eq. 4b having a magnitude less than or equal to one is a fact. The work group may have a magnitude larger than one but it must be ≤ 3, which is on the same order of magnitude of one, i.e., $o(1)$. The only assumption in stating that all of the groups in Equation 4b are $o(1)$ is the assumption that we scaled with the largest term on the left-hand side of Eq. 2, since mathematically dividing a smaller term by a larger term will result in a group magnitude being less than one. Note that this assumption is temporary and will be verified later in the final step of the algebraic scaling procedure being developed.

Now if we want a group to "disappear," we want its magnitude to be very small compared to the magnitude of the Potential Group or for the current case of the Kinetic Group

$$\left|\frac{\Delta v^2}{2(g\Delta z)}\right| \ll 1 \qquad (5)$$

Using a 1% error criterion turns Eq. 5 into

$$\left|\frac{\Delta v^2}{2(g\Delta z)}\right| \le 0.01 \qquad (6)$$

What Eq. 6 states is

$$\frac{\Delta P}{\rho(g\Delta z)} \pm 0.01 + 1 \cong \frac{\Delta P}{\rho(g\Delta z)} + 1 \qquad (7)$$

which is factual to within at least two significant figures or approximately a 1% error. In transport or reaction engineering modeling, it is acceptable to ignore groups if it only incurs a very small (~1%) error.[4] In the context of engineering calculations, ask the students to consider the precision of the value assigned to the last group in Eq. 7, 1.00 ± 0.01, in light of the error propagation in calculations resulting from typical sensor accuracies of equipment such as pressure gauges, elevation monitors, fluid velocity meters, etc. It, therefore, is of value to have the students observe that a 1% error in the reduction-to-practice is, in general, reasonable and acceptable.

Now if we force Eq. 6 to be true, then the Kinetic Group is negligible, and the result from Part a is valid to within a 1% error. So, solving Eq. 6

$$\Delta v^2 \le 0.01|2(g\Delta z)| = 0.02\left(9.81\frac{m^2}{s}\right)(3810m) \qquad (8a)$$

$$v(exit) \le 27.3\frac{m}{s} \qquad (8b)$$

The students should meditate on the fact that the exit velocity is not zero; however, $\Delta v^2/2$ is very small compared to $g\Delta z$. This is the key point: Not zero but very small compared to… Plugging Eq. 8b into Eq. 3 gives the answer of a tunnel/pipe exit diameter of at least 1.87 meters or ≈ 6 feet. Of course, the final diameter might need to be larger after considering friction losses and fluid velocity induced erosion of the tunnel; however, our boss has the requested order-of-magnitude estimate of the minimum required diameter.

The absolute value sign introduced into Eq. 5 was so that the inequality evaluates the magnitude of the group and not its value. Consider that without the absolute value sign, any negative variable in the group would make the group's value less than one even if the group has a magnitude greater than one. In fact, this is the case in Eq. 5 where Δz is negative and without the absolute value sign, any value of $\Delta v^2/2$ would result in a Kinetic Group being less than one.

Quality Assurance/Quality Control (QA/QC) Check

As illustrated in the next Key Takeaways subsection, it is a good idea to confirm proper scaling by checking the absolute values of all the groups in the scaled equation after determining that v(exit) ≤ 27.3 m/s:

- Pressure Group = $\Delta P/[\rho(g\Delta z)] = 0.00100$
- Work Group = $-W_s/[m(g\Delta z)] = 0.999$

So, all of the scaled groups are $o(1)$, passing the QA/QC check.

Note that in this specific case the Pressure Group is negligible being also ≤ 0.01. INTERESTING! In what may be the largest Δz ever proposed for a "Pipe Draining a Tank" problem, with the tank and pipe ends both open to the atmosphere, we can still state that $\Delta P \approx 0$.

Key Takeaways from the Initial Example Problem

First the vocabulary could be tricky, specifically between "variables," "terms," and "groups." Here the text used "groups" to refer to scaled algebraic terms. Both the algebraic terms and groups are made up of variables. The second takeaway is that the scaling process had the added benefit of creating design equations for the exit pipe diameter that did not exist prior to scaling; namely, plugging Eq. 6 into Eq. 3. In other words, it created a solution where one was not possible.

The third takeaway is that "negligible" is relative. The criterion to declare the change in kinetic energy is negligible is that the Kinetic Group was very small (≤ 0.01) not that the fluid velocity was very small. In the above example problem, the fluid velocity increased to 27 m/s, which is 97 km/hr or 60 miles/hr. This velocity is not "small" compared to conventional human experiences. The negligible criterion used in this manuscript (≤ 0.01) is a reasonable rule-of-thumb; however, error criteria other than 1% might be more appropriate, depending on the application.

The fourth takeaway is the following basic steps in the algebraic scaling procedure:

1. Choose the largest actor, i.e., the largest term
2. Divide through by the largest actor
3. Create a new equation by setting the magnitude of the "group" you want to neglect equal to ≤ 0.01
4. Solve for the criterion to neglect the term or variable of interest
5. QA/QC check the scaled groups

Choosing the largest actor may not always be as easy as the elevation change in the above example problem. For example, why did we not choose to divide through by the work term? After all, W_s is on the order of trillion of watts. The case-specific answer is that the work term includes mass flow, which is unknown until the Kinetic Group is forced to be negligible. When it is not self-evident which is the largest actor to choose in Step 1 of the algebraic scaling procedure, consider one or more of the following to avoid a trial-and-error process in selecting the best largest actor:

- What is the primary cause of the process? Water flow in the above example is the process and gravity (potential energy) is the cause.
- Divide through with what must be retained. Here the potential energy is the source of energy converted into work (the goal of the process). Therefore, the primary energy source must be retained in the equation.
- Look for illogical scaling results that stem from not choosing the correct largest actor.

What are illogical scaling results? Consider that if we incorrectly chose to scale with the pressure term, then during the QA/QC check, the incorrectly scaled potential energy group = $\rho(g\Delta z)/\Delta P = 997$. This is not on the order of one, and, therefore, is an indication of incorrect scaling.

KINETIC ENERGY ASSUMPTIONS IN INCOMPRESSIBLE FLUID FLOW

This section is a simple problem for students to practice the basic procedural steps of dividing through by the largest actor and determining the criterion for ignoring a term in an equation. This problem focuses on a common assumption used in draining tank problems; namely, the velocity of the fluid surface in the tank or a dam impoundment is zero. This assumption was made without explanation in the prior example problem. The problem is straightforward enough that after the instructor describes the set-up, the students could solve it during class in small working teams.

Consider the classic draining tank problem, shown in Figure 2, which could be solved using the generalized Bernoulli Equation. The conventional assumption is that the fluid in the tank at Point 1 has a velocity, $v_1 = 0$. However, while v_1 is small, it is obvious from the conservation of mass that the surface of the fluid in the tank must have a downward velocity.

Problem Statement: "What ratio of tank cross sectional area, A_1, to exit pipe cross sectional area, A_2, will allow the assumption that v_1 is negligible?"

The following is the system of equations for this problem statement:

$$\frac{\Delta v^2}{2} = \frac{(v_2)^2}{2} - \frac{(v_1)^2}{2} \tag{9}$$

$$v_i = \frac{G}{A_i} \tag{10}$$

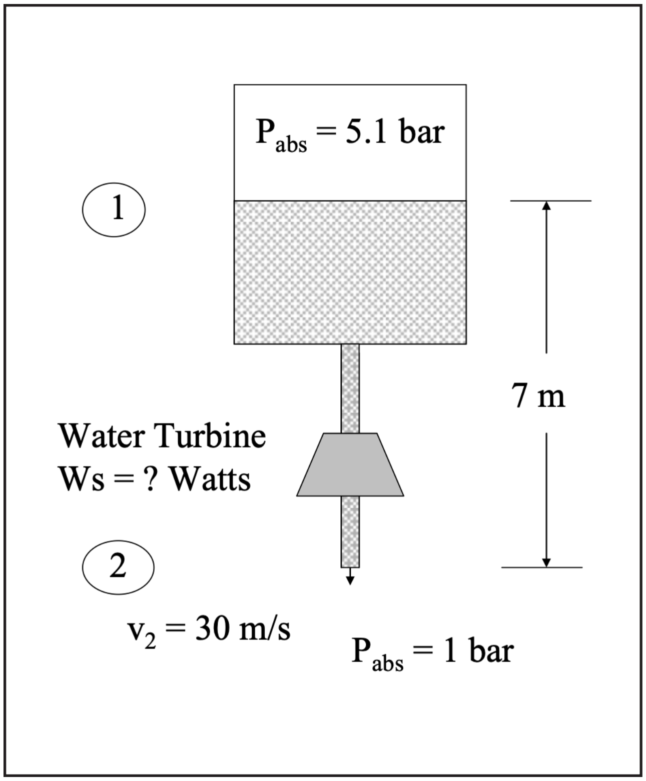

Figure 2. *Diagram for standard kinetic energy assumption that fluid velocity at Point 1 is negligible.*

where v_i is the fluid velocity at location i, G is the volumetric fluid flow, and A_i is the cross-sectional area at location i.

Solution: The velocity at location 2 must be retained and, by inspection $v_2 > v_1$, so divide through with $(v_2)^2/2$ to obtain

$$\frac{\Delta v^2}{(v_2)^2} = 1 - \frac{(v_1)^2}{(v_2)^2} \tag{11}$$

A 1% error criterion for neglecting the velocity at location 1 is

$$\frac{(v_1)^2}{(v_2)^2} \leq 0.01 \tag{12}$$

Plug in Eq. 10 and solve for the cross-section area ratio

$$\frac{A_1}{A_2} \geq 10.0 \tag{13}$$

So, if the cross-sectional areas differ by a factor of 10, one of the location velocities is negligible compared to the other location for the purposes of calculating the kinetic energy change during incompressible fluid flow. This result is not surprising; however, if you assume circular cross sections and calculate the minimum ratio for the diameters, the answer is only 3.16.

STEAM NOZZLE: DETERMINING THE DESIGN SPECIFICATION SO $Q \approx 0$

This example illustrates how to use scaling to determine design specifications. It also could be a situation where the students have limited physical experience, forcing the students to rely on the scaling process instead of their gut feelings. While the previous two examples used incompressible fluid flow, this example will help students to see that the scaling procedure works with other engineering governing equations.

Consider the high-pressure steam nozzle (feed conditions of 1000 kPa at 400 °C) in Figure 3. The steam nozzle converts the enthalpy of the feed steam into kinetic energy and an energy balance, Eq. 14, governs this energy conversion

$$\Delta H + \Delta KE + \Delta PE = Q - W_s \tag{14}$$

where ΔH is the change in steam enthalpy, ΔKE is the change in kinetic energy, ΔPE is the change in potential energy, Q is heat transfer, and W_s is the shaft work. Since the purpose of a steam nozzle is to turn ΔH into ΔKE, engineers design steam nozzles to have negligible amounts of the energy conversion into ΔPE, W_s, and Q. Students can be told that the standard design specification for a steam nozzle is to have $\Delta PE \approx Q \approx W_s \approx 0$, since these represent parasitic energy losses.

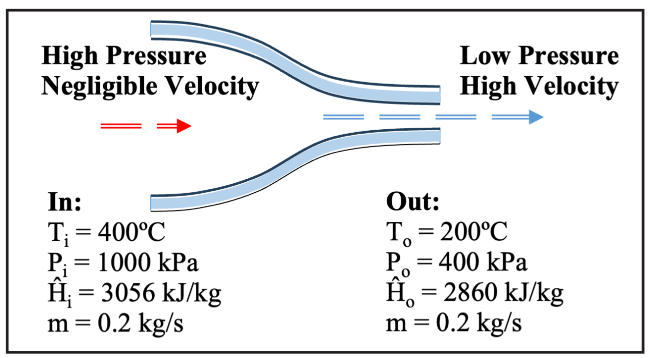

Figure 3. *Adiabatic steam nozzle problem setup.*

Problem Statement: "Find the maximum Q that will allow the assumption of an adiabatic steam nozzle."

In this case, both ΔH and ΔKE must be retained (the purpose of the nozzle is $\Delta H \rightarrow \Delta KE$). W_S and ΔPE, by design, are negligible, so a logical choice for the largest actor is ΔH. Dividing through by ΔH

$$1 + \frac{\Delta KE}{\Delta H} + \frac{\Delta PE}{\Delta H} = \frac{Q}{\Delta H} - \frac{W_s}{\Delta H} \tag{15}$$

A 1% error criterion for neglecting heat transfer is

$$\left|\frac{Q}{\Delta H}\right| \leq 0.01 \qquad (16)$$

Using the data in Figure 3

$$Q \leq 0.01|\Delta H| = 0.01\left(0.2\frac{kg}{s}\right)|2860 - 3264|\frac{kJ}{kg} = 0.808\frac{kJ}{s} \qquad (17)$$

So, the design specification to allow the adiabatic nozzle assumption is to have $Q < 800$ W for this specific nozzle, steam feed rate, feed conditions, and exit conditions. Students should be told that the final answer, $Q < 800$ W, is not a universal design criterion; however, they should focus on the procedure that resulted in the case specific result.

A two-slice bread toaster uses from 700 to 1100 W of power when in use, while an average toaster will use around 900 W. So, holding the steam nozzle, in Figure 3, designed to be "adiabatic" would be the same as holding the red-hot heating elements of a toaster. Since scaling called this "small," would you be willing to hold it without insulated gloves? However, for negligible Q in calculating the velocity of exit steam, it is very small. If, however, your engineering task is energy savings or operational safety, it may not be small. This time a key takeaway is that "small" is not only relative to the other terms in the governing equation but to the context of the larger problem statement. Another key takeaway is that scaling can help in engineering the proper process design specifications.

NEGLECTING CHANGES IN POTENTIAL ENERGY: A HOMEWORK PROBLEM

The previous steam nozzle example ignored changes in potential energy, which is common for processes with large changes in system enthalpy or internal energy, U, such as the open system steam nozzle or piston/cylinder closed system problems.

Consider the piston/cylinder in Figure 4 where a freely moving piston maintains a pressure of 200 kPa on water. Initially the water is liquid at 2 °C, then heat transfer occurs until the piston contains saturated water vapor. This is a classic piston/cylinder problem where students are asked to find the specific work, specific heat transfer, and final temperature, using the following closed system energy balance equation:

$$\Delta U + \Delta KE + \Delta PE = Q - W \qquad (18)$$

where W is the expansion/contraction work. Most conventional problem statements provide no information, such as piston diameter, that would allow calculation of the change is elevation of the piston or the center of gravity of the water between the initial liquid and final steam states of the water. It is standard practice in these problems to assume the change in potential energy is negligible; but, why? Here scaling will illustrate why the standard practice is generally reasonable.

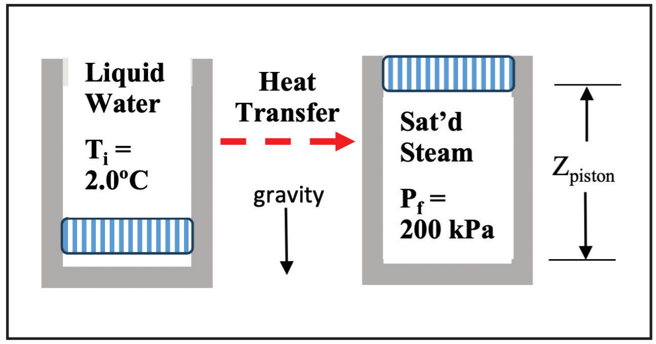

Figure 4. *Piston/cylinder problem diagram.*

Problem Statement: Use $o(1)$ algebraic scaling to estimate what rise in piston elevation, ΔZ_{piston}, would result in a 1% error in the final answers if the solution ignored ΔPE.

In this case ΔU, Q, and W must be retained. Heat transfer, Q, is the source of energy, and its energy is converted into ΔU and W; therefore, Q is the largest actor. Unfortunately, it is also an unknown. ΔU is a state function, and since we know the initial and final states, we do know ΔU. Here choosing ΔU instead of Q makes the problem easier, and, since $\Delta U < Q$, this will underestimate the piston rise, which is a conservative estimate in this situation. Dividing Eq. 18 through by ΔU

$$1 + \frac{\Delta KE}{\Delta U} + \frac{\Delta PE}{\Delta U} = \frac{Q}{\Delta U} - \frac{W}{\Delta U} \qquad (19)$$

A 1% error criterion for neglecting potential energy change is

$$\left|\frac{\Delta PE}{\Delta U}\right| \leq 0.01 \qquad (20)$$

The specific internal energies, Û, of the final and initial states are:

- Û(liquid, 2 °C, 200 kPa) ≈ Û(sat'd liquid, 2 °C) = 8.4 kJ/kg
- Û(sat'd vapor, 200 kPa) = 2529.2 kJ/kg

Plugging into Eq. 20 the specific internal energy values and the definition of ΔPE

$$\left|\frac{mg\Delta Z_{cg}}{m(2529.2 - 8.4)\frac{kJ}{kg}}\right| \leq 0.01 \qquad (21)$$

$$\Delta Z_{cg} \leq 0.01 \frac{(2529.2 - 8.4)\frac{kJ}{kg}}{9.81\frac{m}{s^2}}\left(\frac{1000\,J}{kJ}\right)\left(\frac{kg\,m^2/_{s^2}}{J}\right) = 2570\,m \quad (22)$$

Since ΔZ_{cg} is the change in the center of gravity of the water from liquid to steam, the piston rise would be approximately 2 times ΔZ_{cg} or 5.1 km! Therefore, "negligibly small" is not small is self-evident in this case.

The takeaway here is that the ΔZ_{cg} required for the change in potential energy to have a significant impact on the final answer (> 1 % error) is illogically large and has no physical meaning. After all, if the cylinder contained 1.0 kg-water, the piston/cylinder must have a cross-sectional area of 0.000172 m^2 (or a 1.5 cm diameter) for the piston to rise 5.1 km. Alternatively, a more practical 10-cm diameter cylinder would have an initial liquid water depth of 12.7 cm and a final piston height of only 113 meters, significantly less than 5.1 km. However, the result does illustrate why the change in potential energies are generally ignored when there are large changes in internal energy (or enthalpies).

It is left up to the reader to solve this piston/cylinder problem with ΔPE reasonably set equal to zero and assuming that there was no change in the piston/cylinder velocity between the initial and final states. The solution results in a specific work of 177 kJ/kg and a specific Q of 2700 kJ/kg. Now the quality control check of the scaling results in the following:

- Heat Group = $Q/[\Delta U]$ = 1.07
- Work Group = $W/[\Delta U]$ = 0.0701
- Kinetic Group = 0

So, the groups are all of the $o(1)$, and there is no need to rescale even though we did not technically choose the largest actor to scale with.

DETERMINING WHICH BOUNDARY LAYER CONTROLS MASS TRANSFER

This is an advanced course concept example. Students can use the results in evaluating mass transfer systems. Alternatively, they can use the results in designing experiments for the determination of mass transfer coefficients in process equipment.

Consider mass transport from one phase to another across a phase boundary. For illustrative purposes consider the transport of a mythical chemical, TED, from water to air (Figure 5) where the fluid dynamics determine the values of the two boundary-layer mass transfer coefficients. In contrast to heat transfer, determining the controlling boundary layer for mass transfer does not come from comparing the relative numerical values of the two mass transfer coefficients. It comes from scaling the equation for the overall mass transfer coefficient as illustrated below.

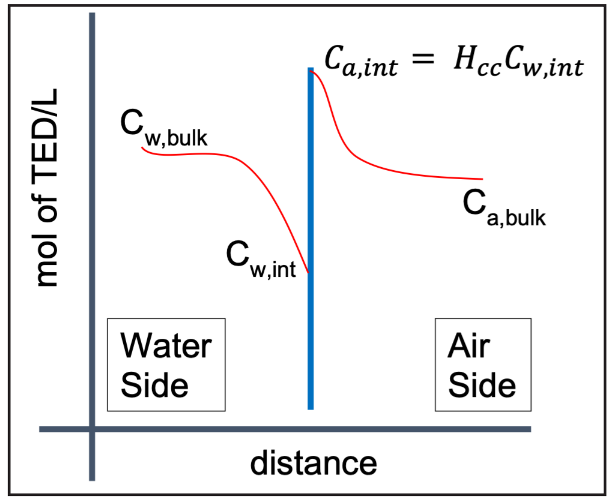

Figure 5. *Illustration of two-phase boundary-layer mass transfer of TED.*

For steady state mass transfer, any of the following four equations define the rate of TED movement from the bulk water phase to the bulk air phase, r in mol-TED/sec:

$$r = -Ak_l\left(C_{w,int} - C_{w,bulk}\right) \quad (23)$$

$$r = -Ak_g\left(C_{a,bulk} - C_{a,int}\right) \quad (24)$$

$$r = -Ak^*_{gl}\left(C_{a,bulk} - H_{cc}C_{w,bulk}\right) \quad (25)$$

$$r = -Ak_{lg}\left(C_{a,bulk}/_{H_{cc}} - C_{w,bulk}\right) \quad (26)$$

where A is the phase interfacial area (cm^2), k_l is the water film mass transfer coefficient (cm-water/sec), k_g is the air film mass transfer coefficient (cm-air/sec), k^*_{gl} is the overall mass transfer coefficient in gas units (cm-air/sec), k_{lg} is the overall mass transfer coefficient in liquid units (cm-water/sec), H_{CC} is a form of the Henry's Law constant (cm^3-water/cm^3-air), $C_{W,j}$ is the concentration of TED in the water at location j (mol-TED/cm^3-water), and $C_{a,j}$ is the concentration of TED in the air at location j (mol-TED/cm^3-air). Subscripts "int" stands for interface and "bulk" stands for bulk fluid.

The following are the relationships between the overall mass transfer coefficients and the single film mass transfer coefficients:

$$\frac{1}{Ak^*_{gl}} = \frac{1}{Ak_g} + \frac{H_{cc}}{Ak_l} \quad (27)$$

$$\frac{1}{Ak_{lg}} = \frac{1}{Ak_l} + \frac{1}{AH_{cc}k_g} \quad (28)$$

If the liquid film controls, then $k_{lg} \approx k_l$. In this case the term $1/(Ak_l)$ in Eq. 28 must be retained and we will divide Eq. 28 with it.

$$\frac{Ak_l}{Ak_{lg}} = 1 + \frac{Ak_l}{AH_{cc}k_g} \tag{29}$$

By inspection of Eq. 29, the 1% error condition for declaring liquid film control is

$$\left|\frac{Ak_l}{AH_{cc}k_g}\right| \leq 0.01 \tag{30}$$

In a rectangular coordinate system, the interfacial areas are equal, and the condition for liquid film control becomes

$$\frac{k_l}{k_g} \leq 0.01H_{cc} \tag{31}$$

Note that the criterion, unlike for heat transfer, is not only the comparison of the film transport coefficients but also includes Henry's law. Physically, the concentrations at the interface, while assumed to be in equilibrium, are not numerically equal (Figure 5). This is in contrast to heat transfer, where the water and air temperatures at the interface are in equilibrium and numerically equal. Also, the direct comparison of the mass transfer coefficient k's would be unit inconsistent since k_l has units of cm-liq/sec and k_g has units of cm-air/sec. This means that the ratio of k's in Eq. 31 is not unitless but has the same units as H_{CC}. Some engineers [10] refer to the ratio in Eq. 31 as the "critical Henry's law constant" because, for a system with these film transport coefficients, the mass transfer resistance of the air film will equal the resistance of the water film for a solute with an $H_{CC} = k_l / k_g$. The key takeaway here is that algebraic scaling revealed a critical system parameter.

It is left up to the reader to scale Eq. 27 to determine when the air film controls the mass transfer.

OTHER ILLUSTRATIONS

To further show that the scaling procedure works with many types of engineering equations, the following examples are presented without detailed solution steps or commentary.

In thermodynamics the definition of enthalpy, ΔH, is

$$\Delta H = \Delta U + \Delta(PV) \tag{32}$$

where P is the absolute pressure and V is the material volume. Consider a situation where Substance-X is melting while undergoing a pressure change. What is the limit on pressure change, ΔP, to allow the assumption that $\Delta H = \Delta U$? Assume the following Substance-X properties: $\Delta H_{melting}$ = 100 kJ/kg and specific volume = 0.002 m^3/kg. Also assume that specific volume is constant with pressure and the solid to liquid phase change. Answer: $\Delta P \leq 5$ bars for a 1% error in the assumption that $\Delta H = \Delta U$.

Adsorption equilibrium can have a variety of isotherm models depending on the surface chemistry. One well known isotherm model is Langmuir

$$q = \frac{q_{max}K_{eq}[C]}{1 + K_{eq}[C]} \tag{33}$$

where q is the loading of the solute on the surface, q_{max} is a model parameter related to the number of adsorption sites on the surface, K_{eq} is a model parameter related to the strength of the solute bonding to the surface, and $[C]$ is the concentration of the solute in the fluid contacting the surface.

Problem Statement: For what concentration range does a Langmuir system behave as a linear isotherm? Answer: $0 < [C] \leq 0.01/K_{eq}$.

Problem Statement: For what concentration range does a Langmuir system behave as an irreversible isotherm? Answer: $[C] \geq 100/K_{eq}$.

Similar to the previous closed piston/cylinder problem, steam turbine problems (i.e., open system energy balance problems) routinely ignore potential energy changes in the calculations. Many textbooks contain steam turbine problems similar to the following rephrased from Felder and Rousseau.[11] Consider a steam turbine generating 70 kW of work from a feed of 500 kg/h of steam entering at 44 atm of pressure and a temperature of 450°C. The steam leaves the turbine 5 meters below its entrance elevation at atmospheric pressure. The heat loss from the turbine is 11.6 kW. The velocities of the steam at the entrance and exit are 60 m/s and 360 m/s, respectively. The student could be asked to determine the significance of ignoring the potential energy change (less than 0.01% change in the final answer) or what elevation change would have a 1% impact on the final work generated (about 0.5 km).

It is left up to the reader to speculate on algebraic scaling examples for determining system design specifications from various engineering disciplines such as allowable heat loss from a reactor (chemical engineering), wire resistance ratings (electrical engineering), or maximum allowable pressure drops in air conditioning ducts (mechanical engineering).

SUMMARY AND COMMENTS

Making assumptions is integral to the practice of engineering.[1-5] Algebraic scaling is a pedagogical tool for teaching students about what is a good assumption by giving insights into when a term is negligible. Scaling helps remove the mystery around why, in general, some terms are assumed to be negligible by the professor. In addition, the algebraic scaling method can reinforce the concept of significant figures. The 1% error criterion for eliminating terms does not alter the final answer for at least two significant figures. For example, this manuscript reported all numerical answers in three significant figures since reporting a specific Q in the piston/cylinder example as 2697 kJ/Kg would be meaningless when the largest group in the scaled algebraic equation had a precision of 1.00 ± 0.01. This, therefore, is another pedagogical element of algebraic scaling — namely, the inaccuracy of reporting answers with more significant figures than the calculation precision supports.

This manuscript also moved the algebraic scaling method beyond a pedagogical tool to an application-oriented tool. In the hydroelectric example, the scaling method created an inequality equation that allowed a solution to an underdefined system of equations. It created a solution where one was not possible before scaling. The steam nozzle example illustrated good engineering design for minimizing parasitic factors. Here, scaling helped in engineering the proper design specifications. Furthermore, the two-film mass transfer example showed that scaling can reveal critical system parameters.

Before concluding, there are two caveats. First, the method described above does not work for differential equations. For senior/graduate student level material, the reader should consult the work of Dr. Krantz for the techniques needed for scaling analysis of differential equations.[2,4,8] Second, an instructor should understand that some students may incorrectly assume that the result of an example problem (i.e., the steam nozzle) is universally applicable. In this case the student assumes that if $Q < 800$ W, the process is adiabatic. This would be an incorrect outcome. The instructor must emphasize that the student must learn the method and not the case-specific final answer.

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