

The Method of Matched Asymptotic Expansions*

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THE METHOD of matched asymptotic expansions is a mathematical technique which deals with the solution of differential equations containing a small parameter ϵ . To begin, let us consider the nonlinear system

$$\frac{dy}{dt} + y + \epsilon y^2 = 0 \quad t \geq 0 \quad (1)$$

with $y = 1$ at $t = 0$. The exact solution is simply

$$t = \ln \frac{1 + \epsilon y}{y(1 + \epsilon)} \quad (2)$$

but, for purposes of this discussion, we shall suppose that this exact answer is not available. Thus, to develop a solution for small ϵ we need to make use of an approximate technique.

Clearly, one logical way of proceeding would be to assume that y is analytic in ϵ and to expand the unknown solution in the series

$$y = \sum_{n=0}^{\infty} \epsilon^n y_n(t), \quad y_0(0) = 1, \quad y_n(0) = 0 \quad n \geq 1, \quad (3)$$

with the functions $y_n(t)$ assumed independent of ϵ . By substituting eq. (3) into eq. (1) and equating like powers of ϵ , we then easily obtain the system of linear equations:

$$\frac{dy_0}{dt} + y_0 = 0, \quad \frac{dy_1}{dt} + y_1 = -y_0^2, \text{ etc.},$$

the first of which yields

$$y_0 = e^{-t} \quad (4)$$

The technique just described is known as a *regular* perturbation expansion. Although ap-

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plicable to a number of problems for which eq. (3) converges for ϵ sufficiently small, regular perturbation solutions are rarely very exciting from a practical point of view because here the effects of the terms containing ϵ are small everywhere provided that $\epsilon \ll 1$. Hence, problems that are amenable to this technique can generally be simplified without much loss of accuracy by merely setting $\epsilon = 0$ in the basic equation and solving what is left. For example, for $\epsilon = 0.1$, values of y computed from eq. (4), which satisfies eq. (1) with $\epsilon = 0$, are within about 10% of those calculated from the exact solution, eq. (2), throughout the interval $0 \leq t < \infty$.

On the other hand, let us take a look at the system

$$\epsilon \frac{dy}{dt} + y = e^{-t}, \quad t \geq 0 \quad (5)$$

with $y = 0$ at $t = 0$, the exact solution to which is

$$y = \frac{1}{1 - \epsilon} \{ e^{-t} - e^{-t/\epsilon} \} \quad (6)$$

Here, a regular perturbation expansion will fail miserably for, if we proceed as before and make use of eq. (3), we obtain not eq. (6), but rather

$$\tilde{y} = (1 + \epsilon + \epsilon^2 + \dots) e^{-t} = \frac{1}{1 - \epsilon} e^{-t} \quad (7)$$

which is identical to the first term of the exact solution. We can clearly see then that, in this simple example, *no matter how small the value of ϵ there will always exist an interval $0 \leq t \leq O(\epsilon)$, within which the true solution and that derived from a regular perturbation series will differ by $O(1)$* . In particular, had we been

interested in the slope at $t = 0$, the use of eq. (7) would have given us a value of $-1/(1 - \epsilon)$, in total disagreement with the exact answer $1/\epsilon$.

THE FAILURE OF the regular perturbation expansion, when applied to the second example, is of course not surprising since, by substituting eq. (3) into eq. (5), we have effectively reduced a first order ordinary differential equation into a set of algebraic equations the solution to which is independent of the imposed initial condition. Hence, the regular perturbation series cannot represent in general a *uniformly valid approximation* to the true solution even as $\epsilon \rightarrow 0$, and must in fact cease to apply when the origin, $t = 0$, is approached. Moreover, in contrast to our first problem, we can see that here the term containing ϵ is no longer of secondary importance, in that its presence has an essential bearing on the form of the solution near the origin.

It is then with problems of the second type, which are not amenable to the regular perturbation expansion, that the method of matched asymptotic expansions is concerned. Its basic features are as follows:

Again, returning to eq. (5), we recognize that, near the origin, the term $\epsilon dy/dt$ must be retained in the differential equation if the solution y is to satisfy the initial condition. This means that, for $\epsilon \rightarrow 0$, the term dy/dt must become $O(1/\epsilon)$ near $t = 0$, if the product $\epsilon dy/dt$ is to remain finite. In turn, this suggests the substitution $\tau \equiv t/\epsilon$, in terms of which eq. (5) becomes

$$\frac{dy}{dt} + y = e^{-\epsilon t} = 1 - \epsilon t + \frac{1}{2}(\epsilon t)^2 + \dots \quad (8)$$

Thus, provided that ϵt (or, conversely, t) is small, we can obtain a solution to eq. (8) of the form

$$\bar{y} = \sum_{n=0}^{\infty} \epsilon^n \bar{y}_n(\tau) \quad (9)$$

in which the individual terms $\bar{y}_n(\tau)$ are now functions of τ and do not depend explicitly on ϵ . Substitution of eq. (9) into eq. (8), together with the initial condition, results then in

$$\bar{y} = 1 - e^{-\tau} - \epsilon(\tau - 1 + e^{-\tau}) + \dots \quad (10)$$

Thus, for the problem at hand, we have developed two solutions, eqs. (7) and (10), which hold, respectively, within the two different ranges of the variable t , $t > O(\epsilon)$, and $t \leq O(1)$. Of these, eq. (7) termed the "outer" solution behaves properly as $t \rightarrow \infty$, whereas the "inner" solution, eq. (10), satisfies the initial condition at $t = 0$.

In addition, since eqs. (7) and (10) are, presumably, different expansions in ϵ of the same function, i.e. the exact solution, they should match in some overlap region in which both expansions are valid, provided of course that such an overlap region exists. In the present problem, for example, we require that the "outer" solution, as $t \rightarrow 0$, match with the "inner" solution, as $\tau \rightarrow \infty$, (note that for any given $t < 1$, τ can be made arbitrarily large by taking ϵ sufficiently small!) up to the order of the approximation. To demonstrate this, we expand the outer solution in t and ϵ so that, up to and including terms of $O(\epsilon)$,

$$\begin{aligned} \bar{y} &= (1 + \epsilon \dots) (1 - t + \dots) \\ &= 1 + \epsilon(1 - \tau) + O(\epsilon^2) \end{aligned}$$

which, to $O(\epsilon^2)$, is seen to agree exactly with the limit of eq. (10) as $\tau \rightarrow \infty$. In fact, by appropriately expanding the two solutions, we can easily demonstrate that these two will match exactly up to all orders in ϵ .

Before turning to a few examples of physical interest, let us consider briefly the slightly more complicated system

$$\epsilon \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0 \quad t > 0 \quad (11)$$

with initial conditions $y = 0$, $dy/dt = 1$ at $t = 0$. Here, the first term of the outer solution is obviously

$$\bar{y} = Ae^{-t} \quad (12)$$

where the constant of integration A remains, for the moment, unspecified. (Note that it would have been erroneous to determine A from either initial condition since the origin $t = 0$ is located outside the region of validity of the outer solution, eq. (12)!). The inner solutions on the other hand, which of course must satisfy both initial conditions, is obtained by noting that near $t = 0$, d^2y/dt^2 is $O(1/\epsilon)$, dy/dt is $O(1)$ and y is $O(\epsilon)$, so that, with $\tau \equiv t/\epsilon$,

$$\bar{y} = \epsilon \{ 1 - e^{-\tau} \} + O(\epsilon^2)$$

The two solutions will then match, in the sense described above, if A in eq. (12) is set equal to ϵ .

This last example is meant to illustrate one very important and useful property of the matching requirement. That is, in addition to providing us with some confidence regarding the validity of the expansions, the matching condition allows us to determine uniquely the arbi-

trary constants that generally appear in both solutions. Admittedly, this is not clear from the previous two examples, but, in general, an outer solution will contain arbitrary constants because it cannot satisfy any imposed conditions at $t = 0$, whereas an inner solution will also involve adjustable constants, since it cannot be applied at $t = \infty$; thus, the fact that the two solutions are required to join smoothly within the overlap region provides us then with the additional restrictions from which these constants can be evaluated unambiguously.

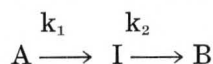
APPLICATIONS

THE METHOD, as outlined above, has found numerous applications in various areas of chemical engineering, but, for the purposes of this discussion, we shall consider briefly only three physical examples which illustrate different aspects of the technique.

I. Improving the Quasi-Steady State Approximation in Chemical Kinetics.

The quasi-steady state approximation (QSSA) has occupied a prominent role in chemical kinetics ever since its inception by Bodenstein in 1924. As is well-known, its use can often result in a substantial simplification of the overall kinetic expression without much loss in accuracy. Nevertheless, there are a number of problems requiring a refinement to the QSSA. This can be accomplished in a straightforward manner using the methods of matched asymptotic expansions (1).

Consider the simple kinetic scheme



in which A is the reactant, B the product, and I the intermediate. Let $[A]$ be the concentration of A with $[A]_0$ being its initial value. Then, if

$$z \equiv \frac{[A]}{[A]_0}; \quad y \equiv \frac{[I]}{\epsilon[A]_0}; \quad \epsilon \equiv \frac{k_1}{k_2}; \quad \theta \equiv k_1 t$$

with t being the time, the basic equations become

$$\frac{dz}{d\theta} = -z; \quad z(0) = 1$$

and

$$\epsilon \frac{dy}{d\theta} = z - y; \quad y(0) = 0$$

Clearly, $z = e^{-\theta}$ and, therefore,

$$\epsilon \frac{dy}{d\theta} + y = e^{-\theta}; \quad y(0) = 0$$

which is seen to be identical with eq. (5). Here, the QSSA is given by $y = e^{-\theta}$ which, as shown above, can easily be refined by appropriate expansion in ϵ .

In fact, as shown in references (1 and 2), this method of matched asymptotic expansions can be applied with ease to systems having a large variety of kinetic schemes, thereby extending still further the usefulness of the QSSA.

II. Low Reynolds Number Flow Past Stationary Objects.

A classical problem in fluid mechanics is to determine the drag on a stationary object which is immersed in a uniform stream. In principle, this force can be computed from the solution of the Navier-Stokes equations

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{R} \nabla^2 \mathbf{u} \quad (13)$$

$\mathbf{u} = \mathbf{i}$ at infinity, $\mathbf{u} = 0$ on the surface of the object, where R is the appropriate Reynolds number. However, in view of the non-linearity of eq. (13), an exact solution is generally impossible to obtain unless R is either very large or very small.

For $R \rightarrow 0$ it appears logical to neglect the inertia terms in eq. (13), thus reducing the system to the *linear* equation

$$\hat{\nabla} p = \nabla^2 \mathbf{u}, \quad \hat{p} = pR \quad (14)$$

for which general methods of solution are available (3). In fact, for objects of finite size, the solution of eq. (14), $\mathbf{u} = \mathbf{u}_0$, results in an expression for the drag coefficient which is in agreement with experimental measurements as long as R is sufficiently small; but, any attempt to construct a solution to eq. (13) by means of a regular perturbation expansion of the form

$$\mathbf{u} = \mathbf{u}_0 + R \mathbf{u}_1 + \dots \quad (15)$$

is doomed to failure, because, far from the object, the "Stokes" solution \mathbf{u}_0 no longer represents a uniformly valid approximation to the true solution \mathbf{u} . Therefore, as shown by Proudman and Pearson (4), the small R solution of eq. (13) must be constructed within the framework of the method of matched asymptotic expansions in a way such that an inner solution, eq. (15), is joined properly to the solution of an appropriate "outer" equation in which both the inertia and viscous

terms are of comparable magnitude. A feature of interest here is that the proper inner expansion contains logarithmic terms in R so that, for flow past a solid sphere, the expression for the drag coefficient becomes

$$C_D = \frac{6\pi}{R} \left(1 + \frac{3}{8} R + \frac{9}{40} R^2 \ln R + O(R^2) \right)$$

in which the first term corresponds to Stokes' law.

A qualitatively similar result also applies for the analogous problem of heat transfer from an isothermal sphere to a fluid in Stokes flow (5).

III. Laminar Boundary Layer Theory.

The goal of laminar boundary layer theory is to describe laminar flow at high values of the Reynolds number R in which the viscous terms of eq. (13) are generally negligible except in regions of high shear. To derive the boundary layer equations for a two-dimensional flow we proceed as follows:

Let ω be the component of vorticity in a direction normal to the plane of flow. The equations of motion can then be expressed as (6)

$$\mathbf{u} \cdot \nabla \omega = \frac{1}{R} \nabla^2 \omega,$$

so that, if the viscous terms are neglected in the limit $R \rightarrow \infty$,

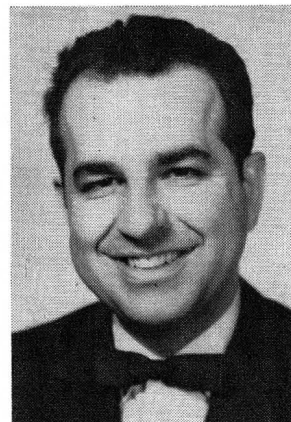
$$\mathbf{u} \cdot \nabla \omega = 0, \text{ which implies } \omega = \omega(\psi)$$

where ψ is the streamfunction. In other words, in an inviscid region the vorticity ω remains constant along a streamline, but, since for a stationary solid in a uniform stream, all the streamlines originate at infinity where $\omega = 0$, we can immediately conclude that the inviscid or outer region must be everywhere irrotational. In view of what has been said in connection with eq. (5), however, we would expect the irrotational solution to break down near the solid surface since, owing to the production of vorticity, ω cannot vanish identically near the solid walls. Hence, there must exist an inner region near the surface where $\frac{1}{R} \nabla^2 \omega$ is $O(1)$, which in turn leads to the result that the thickness of the boundary layer must be $O(R^{-1/2})$. From here on, it is an easy matter to complete the derivation of the boundary layer equations (6) which, as is well known, are essential for determining rates of momentum, heat and mass transfer in laminar high Reynolds number flows.

This is admittedly a very incomplete presentation of the method of matched asymptotic expansions which is described in much more comprehensive detail in references (1, 2, 4, and 5) as well as in a recent book (7). It is hoped, though, that the examples which have been presented above have served to illustrate some of the salient features of this technique, which is quickly beginning to emerge as a mathematical tool of great power, versatility, and usefulness in many diverse areas of chemical engineering.

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