TRANSPORT PHENOMENA EQUATIONS OF CHANGE

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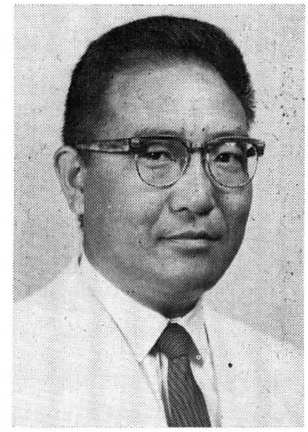

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His interests include heterogeneous catalysis, reactions kinetics, solid state and surface physics, thermodynamics, transport phenomena and energy conversion.

Equations describing fluid motion and energy transport have been derived either from the Eulerian point of view of a stationary fluid element of infinitesimal volume or from the Lagrangian point of view of a macroscopic volume of fluid in motion. In the former derivations, lengthy mass, momentum, and energy balances are involved. In the latter derivations, integral transformation theorems and the Reynold's transport theorem are needed.[1,2] The transition from Newtonian body mechanics to fluids mechanics is less than direct in both of the two derivations.

This note presents a derivation of equations of fluid motion and energy transport by considering an infinitesimal fluid element. δV, **in motion.** In addition to formalistic simplicity, the derivation exposes the conceptual continuity from the Newtonian equation of "body" motion to the continuum motion of fluids.

I. THE RATE EQUATION OF VOLUME DILATION

Let the mass velocity of an infinitesimal volume element δV be **v**. The rate of dilation of δV spanned by the vector **v** is

$$\frac{D}{Dt}(\delta V) = \int\int (\mathbf{v} \cdot \mathbf{n})\, dS \qquad (1)$$

where dS is a surface element, **n** is a unit vector normal to dS. The integration is to be carried out at time t, over all the surface of δV, whose coordinates x_j are equal to $x_j(t)$ with $j = 1, 2, 3$. By the divergent theorem,[2] one has

$$\nabla \cdot \mathbf{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \int\int (\mathbf{v} \cdot \mathbf{n})\, dS \qquad (2)$$

Hence equation (1) can be written as

$$\frac{D}{Dt}(\delta V) = \delta V\, \nabla \cdot \mathbf{v} \qquad (3)$$

equation (2) expresses directly that for an incompressible fluid

$$\nabla \cdot \mathbf{v} = 0 \qquad (4)$$

II. THE EQUATION OF CONTINUITY

The equation of continuity expresses the concept that δV is a closed system as to mass transfer; i.e., a "body." Let ρ denote the density of fluid, the law of mass-conservation gives

$$\frac{D}{Dt}(\rho \delta V) = 0 \qquad (5)$$

Remembering $\delta V \neq 0$, differentiating equation (5) and combining it with equation (3), we obtain

$$\frac{D\rho}{Dt} + \rho\, \nabla \cdot \mathbf{v} = 0 \qquad (6)$$

III. NEWTON'S EQUATION OF FLUID MOTION

In view of equation (5), we can regard δV as a "body" with mass $\rho\delta V$. Applying Newton's second law of motion to the "body" we obtain

$$\frac{D}{Dt}(\rho \delta V \mathbf{v}) = -\int\int \rho\, \mathbf{n}\, dS - \int\int (\mathbf{n} \cdot \tau)\, dS - \rho \delta V\, \nabla \phi \qquad (7)$$

On the left hand side of equation (7) is the rate of change of linear momentum. On the right hand side the first and the second terms are summation of forces acting on the "body" along the inward normal of its surface due to pressure and viscous tensors respectively. The last term is a force acting on the "body" with mass $\rho\delta V$ due to potential field ϕ in energy per unit mass. Noting equation (5), we can rearrange equation (7) in the form

$$\rho \frac{D}{Dt}(v) = -\frac{1}{\delta V}\iint p\,\mathbf{n}\,dS - \frac{1}{\delta V}\iint (\mathbf{n}\cdot\tau)\,dS - \rho\nabla\phi \quad (8)$$

Applying the divergent theorem and noting δV is infinitesimal, we obtain

$$\rho \frac{D}{Dt}(v) = -\nabla - (\nabla\cdot\tau) - \rho\nabla\phi \quad (9)$$

Application of the integral divergent theorem to a tensor τ is infrequent in textbooks, but its proof is not difficult[2-3].

IV. THE ENERGY TRANSPORT EQUATIONS

The kinetic energy transport equation can be directly obtained from equation (9) by noting

$$\mathbf{v}\cdot[\rho\frac{D}{Dt}(\mathbf{v})] = \rho\frac{D}{Dt}(\tfrac{1}{2}|\mathbf{v}|^2) \quad (10)$$

Hence the transport equation of kinetic energy is

$$\rho\frac{D}{Dt}(\tfrac{1}{2}|\mathbf{v}|^2) = -\mathbf{v}\cdot\nabla\rho - \mathbf{v}\cdot(\nabla\cdot\tau) - \rho\,\mathbf{v}\cdot\nabla\phi \quad (11)$$

For the transport equation of potential energy, one notes that ϕ is a scalar point function, therefore

$$\frac{D}{Dt}\phi = \frac{\partial}{\partial t}\phi + \mathbf{v}\cdot\nabla\phi \quad (12)$$

For an energy conserving potential field, (e.g., the gravitational field), ϕ does not depend on time explicitly. Equation (12) becomes

$$\frac{D\phi}{Dt} = \mathbf{v}\cdot\nabla\phi \quad (13)$$

Multiplying equation (13) by ρ, we obtain the transport equation of potential energy

$$\rho\frac{D\phi}{Dt} = \rho\,\mathbf{v}\cdot\nabla\phi \quad (14)$$

Combining equations (11) and (14) we obtain

$$\rho\frac{D}{Dt}(\tfrac{1}{2}|\mathbf{v}|^2 + \phi) = -\mathbf{v}\cdot\nabla\rho - \mathbf{v}\cdot(\nabla\cdot\tau) \quad (15)$$

Now the total energy per **unit mass** consists of kinetic energy, potential energy and internal energy, U, (per unit mass). This transport equation of total energy can be obtained by an over all energy balance on the fluid element δV

$$-\frac{D}{Dt}[\rho\delta V(\tfrac{1}{2}|\mathbf{v}|^2 + \phi + U)] = \iint \mathbf{v}\cdot(\mathbf{n}\,\rho)\,dS + \iint \mathbf{v}\cdot(\mathbf{n}\cdot\tau)\,dS + \iint \mathbf{q}\cdot\mathbf{n}\,dS \quad (16)$$

where the vector $\mathbf{q}$ denotes the **rate of energy dissipation** per unit surface area of all forms of energy including heat flux as a major form. The L.H.S. of equation (16) is the rate of decrease of total energy. The R.H.S. of equation (16) are respectively rate of work done by δV against the pressure, rate of work done by δV against the viscous friction and the rate of energy dissipation as heat. Upon differentiation and combining with equation (5) and then applying the integral divergent theorem with δV approaching to zero, we obtain

$$\rho\frac{D}{Dt}(\tfrac{1}{2}|\mathbf{v}|^2 + \phi + U) = -\nabla\cdot(\rho\mathbf{v}) - \nabla\cdot(\tau\cdot\mathbf{v}) - \nabla\cdot\mathbf{q} \quad (17)$$

Since τ is a symetric tensor, it can be shown that

$$\nabla\cdot[\tau\cdot\mathbf{v}] = \mathbf{v}\cdot(\nabla\cdot\tau) + (\tau\cdot\nabla\mathbf{v}) \quad (18)$$

Hence upon combining equations (15) and (17), we obtain the transport equation for internal energy

$$\rho\frac{DU}{Dt} = [-(\tau\cdot\nabla\mathbf{v}) - \nabla\cdot\mathbf{q}] - \rho(\nabla\cdot\mathbf{v}) \quad (19)$$

The terms in the square bracket are rate of heat generation due to friction and rate of energy transfer to the system mainly as heat respectively. Consequently upon multiplication of equation (19) by $(\delta V\cdot\Delta t)$, it becomes of the form

$$\delta U = \delta Q - \rho\delta V \quad (20)$$

This equation is the familiar first law of thermodynamics for a closed system (i.e., a "body"). In view of the assumption leading to equation (5), equation (20) confirms the self-consistency of the derivation.

NOTATION

v	Mass velocity of fluid
δV	Volume of an infinitesimal fluid element
dS	An infinitesimal surface element
n	A unit vector normal to dS
$\frac{D}{Dt}$	Substantial derivative operator
∇	Del operator $= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$
ρ	Density of fluid, a point function of x_1, x_2, x_3 and time t
ϕ	A scalar potential function of x_1, x_2, x_3
τ	The viscous tensor of a fluid
q	Vector heat energy flux
v	Magnitude of fluid velocity

REFERENCES

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2. Sommerfeld, A. Mechanics of Deformable Bodies, Academic Press, London, 1964, Aris, R. Vectors, Tensors and Basic Equations of Fluid Mechanics, Prentice Hall, New Jersey, 1962, Chapters 3-6.
3. Spiegel, N. R., Vector Analysis, Schaum Co. New York, 1959, pp. 122-3, also Reddick H. W., and F. H. Miller, Adv. Math. for Engineers, John Wiley & Sons, New York, 1955, 3rd ed., pp. 350-4.

ChE book reviews

An Introduction to the Engineering Research Project
Hilbert Schenck, Jr.
McGraw-Hill Book Co.,
New York (1969)

After having directed many theses and overseen thesis direction for many years, this writer has thoroughly enjoyed reviewing this small (178 pg. 5″ x 8″) book.

Intended to be an introduction to the engineering research project, it moves swiftly from the selection of a topic through all the major steps to an expected acceptance of a finished manuscript for publication. The author is relentless as he points out the foibles of faculty and academic systems and is no less discerning as he analyzes student "hang-ups" which would hinder the choice and early completion of a desirable research job. The book is written in contemporary style and should be comprehensible to both the would-be-researcher and his director.

Analyzing the volume in more detail, the reviewer believes that "The Selection of a Project" covers the field well but probably ascribes somewhat more than a normal amount of initiative to a student. Unfortuately the conception of a project more frequently falls on a faculty member than on a student and therefore Chapter 2, "Sources for Project Ideas" (25 pages), is far too long. However the short and meaty "Project Check Sheet" should be noted by everyone.

The chapter "Searching the Literature" attacks the subject with clarity, vigor, and decision. It quickly covers the usual but needed generalizations but follows them up with a well conceived and highly possible case history.

How many times have projects failed for lack of apparatus, time, or cost planning? Here is an author who believes in these efforts as an integral part of the project. Indeed he stresses these activities not only as highly desirable but even mandatory if a real researcher and a satisfactory project are to be produced. His tips are pertinent, timely and frequently annoyingly discerning. Unfortunately the author chooses to elaborate next on his categories of research — an area which he could better have omitted — for although his discussion of "Digital Computer Studies" is a good short approach to a long problem, his "Pedagogical Studies" and "Design and Systems Areas" are far below his overall standards.

In his last two chapters on "Reports" and "Journal Papers and Meeting Presentations" the author has been appropriately and pleasantly brief. He has obviously called upon many experiences, both sad and glad, and has extracted an essence which combines philosophy with practicability.

There is much in this book for new researchers to learn before sad experiences can dishearten or even remove them completely from the field, but the book also may be a gage for a more experienced researcher or research director to recheck his effectiveness.

Surprisingly despite the "heavy" material contained in this book, the style is light, friendly and interesting; it is to be hoped that the experimental project reports will be, too!

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