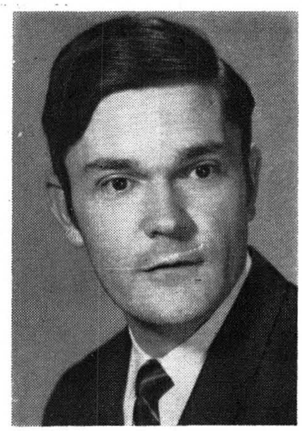
APPLICATION OF PERTURBATION TECHNIQUES TO ANALOG COMPUTATIONS

JOSEPH F. PAUL
California State University
Northridge, California 91324

The object of this study is to investigate the use of perturbation techniques for determining the effect of small changes in parameters on the response of a system. Perturbation techniques have been applied to analog computation for many years (see references). However, an attempt will be made here to present a comprehensive tutorial development of the theory and then illustrate its use by applying it to a particular problem. The problem is studied in detail by first developing the perturbation equations and solving them on the analog computer, and then making an error analysis by solving the equations on the digital computer. A summary of the results is given at the end.



Joseph F. Paul received his BS and MS degrees in ChE from Washington University and has obtained advanced study at University of Southern California. He formerly was employed by Electronic Associates, Inc., TRW Computer Co., and Monsanto Chemical Co. His areas of interest include applied mathematics, computer science, and process control.

PERTURBATION EQUATIONS

THERE ARE TWO METHODS for developing the perturbation equations, both of which yield the same result. They are called the incremental method and the Taylor series expansion. The second method is considered here. If we expand the function Z=f(x,y) in a Taylor series and neglect terms of second order and higher, we obtain:

$$f(x_0 + \Delta x,\ y_0 + \Delta y) = f(x_0,\ y_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,\ y_0} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{x_0,\ y_0} \Delta y \qquad (1)$$

For a particular example let Z=XY. Note that for this case:

$$\left.\frac{\partial Z}{\partial X}\right|_{X_0,\ Y_0} = Y_0;\ \left.\frac{\partial Z}{\partial Y}\right|_{X_0,\ Y_0} = X_0 \qquad (2,\ 3)$$

therefore:

$$Z_0 + \Delta Z = X_0 Y_0 + Y_0 \Delta X + X_0 \Delta Y \qquad (4)$$

where the error would be the sum of the higher order terms. Breaking equation (4) into two parts

$$Z_0 = X_0 Y_0;\ \Delta Z = Y_0 \Delta X + X_0 \Delta Y \qquad (5,\ 6)$$

This is the same result which can be obtained by the incremental method.

INVESTIGATION

AN INVESTIGATION WAS MADE on a particular system which is described below. The object was to determine the response of the system to small changes in one of the parameters using perturbation techniques. First of all the perturbation equation was developed for the system using the incremental method. A study was then made using both the analog and digital computers.

For the first phase of the study both the nominal and perturbation equations were mechanized for the analog computer. A total of five runs were made utilizing increments of 1%, 2%, 3%, 4%, and 5% in the system parameter which was the mass flow rate of fluid in the system. Both the nominal and perturbed values of temperature were recorded on the X-Y plotter.

For the second phase, the system equations were programmed in Hytran Simulation Language, (HSL). This is a digital simulation language which permits the direct programming of dynamic systems in block diagram format. The integration scheme employed was fourth order Runge-Kutta. First the nominal equation was solved to give the steady state temperature, and then the incremental temperature was obtained by solving the perturbation equation and also by perturbing the original equation and subtracting the nominal value from it. This last value was taken to be correct since it was computed from the original equation to eight significant figures, and compared to that obtained from the perturbation equation. The percent error was computed and compared to that predicted by evaluating the error term in the perturbation equation. These results are summarized later in a table.

PERTURBATION PROBLEM

A jacketed tank with constant level control is being heated by steam. How long will it take to reach its steady state temperature and what will it be? What is the effect of a small change in flow rate from the prescribed flow rate?

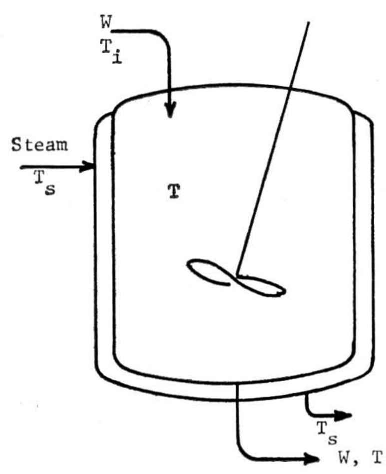


Figure 1

A = area for heat transfer, ft²
V = volume, ft³
U = heat transfer coefficient, Btu/hr-F°-ft²
ρ = density in lbs/ft³
C_p = specific heat in Btu/F°-lb
T = temperature of contents of vessel, F°
W = mass flow rate in lb/hr
T_i = inlet temperature, F°
T_s = steam temperature, F°

The overall equation of the process is

$$\frac{dT}{dt} = \left(\frac{UA}{\rho C_p V}\right)(T_s - T) - \left(\frac{W}{\rho V}\right)(T - T_i) \tag{7}$$

The flow rate W is perturbed such that $W = W_o + \Delta W$ and as a result, T changes by ΔT, i.e., $T = T_o + \Delta T$ and

$$\frac{d(T_0 + \Delta T)}{dt} = \left(\frac{UA}{\rho C_p V}\right)(T_S - T_0 - \Delta T) - \left(\frac{W_0 + \Delta W}{\rho V}\right)(T_0 + \Delta T - T_i) \tag{8}$$

$$\frac{dT_0}{dt} + \frac{d\Delta T}{dt} = K_i(T_S - T_0) - K_i\Delta T - \frac{W_0T_0 - W_o - T_i}{K_2} - \frac{W_0\Delta T + \Delta W T_0}{K_2} + \frac{\Delta W T_i}{K_2} \tag{9}$$

where: $U = 150$ Btu/F-hr-ft^2; $A = 7.5$ ft^2; $\rho = 80$ lb/ft^3;

$C_p = 0.8$ Btu/F°-lb; $V = 100$ ft^3; $K_i = \frac{UA}{\rho C_p V} = 0.176$ hr^{-1};

$K_2 = \rho V = 8{,}000$ lb

Equation (9) can be thought of as the sum of the nominal values and the incremental values. It can be separated into two equations:

$$\frac{dT_0}{dt} = K_i(T_S - T_0) - \frac{(W_0T_0 - W_0T_i)}{K_2} \tag{10}$$

$$\frac{d\Delta T}{dt} = - K_i\Delta T - \frac{(W_0\Delta T + \Delta W T_0)}{K_2} + \frac{\Delta W T_i}{K_2} \tag{11}$$

The same equations can be derived by the Taylor series expansion.

$$F = \frac{dT}{dt} = \left(\frac{UA}{\rho C_p V}\right)(T_S - T) - \frac{W}{\rho V}(T - T_i) \tag{12}$$

$$\left.\frac{\partial F}{\partial W}\right|_{T_0,\, W_0} = - \left(\frac{T_0 - T_i}{\rho V}\right); \quad \left.\frac{\partial F}{\partial T}\right|_{T_0,\, W_0} = - \frac{UA}{\rho C_p V} - \frac{W_0}{\rho V} \tag{13, 14}$$

The nominal equation becomes:

$$\frac{dT_0}{dt} = \left(\frac{UA}{\rho C_p V}\right)(T_S - T_0) - \frac{W_0}{\rho V}(T_0 - T_i) \tag{15}$$

and the incremental equation becomes:

$$\frac{d\Delta T}{dt} = - \left(\frac{T_0 - T_i}{\rho V}\right)\Delta W - \left(\frac{UA}{\rho C_p V} + \frac{W_0}{\rho V}\right)\Delta T \tag{16}$$

These equations agree exactly with equations (10) and (11) and thus verify the equivalency of the two methods.

TABLE 1. SCALED VARIABLES

VARIABLE	MAX	SCALE FACTOR	SCALED VARIABLE
T_0	250	1/250	$T_0/250$
T_S	250	1/250	$T_S/250$
$(T_S - T_0)$	250	1/250	$(T_S - T_0)/100$
T_i	100	1/100	$T_i/250$
$(T_0 - T_i)$	100	1/100	$(T_0 - T_i)/100$
ΔT	1	1	ΔT

NOTE that ΔT is scaled to a maximum of one. Although there is a five per cent change in the parameter ΔW, there is less than one degree change in temperature. The parameter ΔW does not have to be scaled since it does not appear as the output of an amplifier.

SCALED EQUATIONS

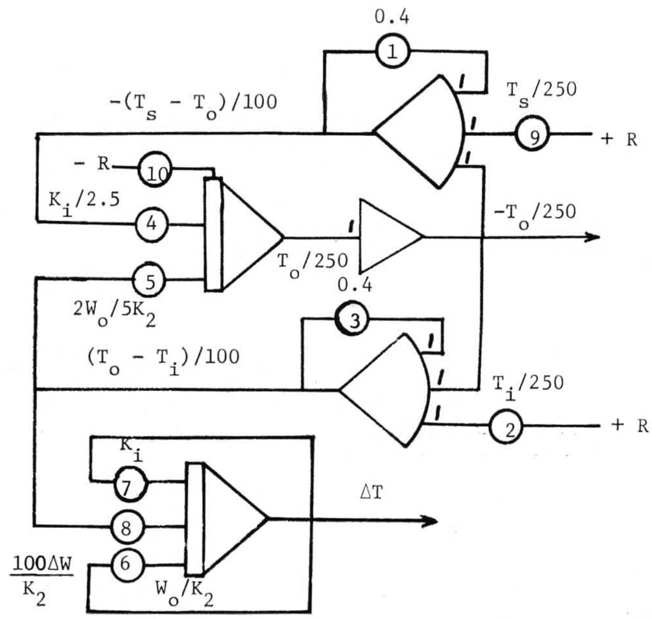
$$\frac{d\left(\frac{T_0}{250}\right)}{dt} = \frac{K_i}{2.5}\left(\frac{T_S - T_0}{100}\right) - \frac{W_0}{2.5K_2}\left(\frac{T_0 - T_i}{100}\right) \tag{17}$$

$$\frac{d(\Delta T)}{dt} = - K_i(\Delta T) - \frac{W_0}{K_2}(\Delta T) - \frac{100\Delta W}{K_2}\left(\frac{T_0 - T_i}{100}\right) \tag{18}$$

$$\left(\frac{T_S - T_0}{100}\right) = 2.5\left(\frac{T_S}{250}\right) - 2.5\left(\frac{T_0}{250}\right) \tag{19}$$

$$\left(\frac{T_0 - T_i}{100}\right) = 2.5\left(\frac{T_0}{250}\right) - 2.5\left(\frac{T_i}{250}\right) \tag{20}$$

Equations (17), (18), (19), and (20) are mechanized as depicted by Figure 2.



ANALOG DIAGRAM
Figure 2

In addition to the present problem, for an illustration of a non-linear set of equations let us put a valve on the inlet flow W still assuming an adequate level control at the outlet. Thus:

$$P_i - P_a = R_v W^2 = P_d \tag{21}$$

where P_i is upstream inlet pressure; P_a is atmospheric pressure; and R_v is resistance coefficient of the value.

$$P_{do} + \Delta P_d = R_v\left(W_o + \Delta W\right)^2 \approx R_v W_o^2 + 2R_v W_o \Delta W \tag{22}$$

$$P_{do} = R_v W_o^2 \text{ or } W_o = \sqrt{\frac{P_{do}}{R_v}}$$

$$\Delta P_d = 2R_v W_o \Delta W \text{ or } \Delta W = \frac{\Delta P_d}{2R_v W_o}$$

$$W_o = 1000 \text{ lb/hr};\ P_{do} = 15 \text{ psi};\ \text{then } R_v = 15 \times 10^{-6}$$

$$\Delta W = \Delta P_d/(2 \times 15 \times 10^{-6} \times 1 \times 10^3) = \Delta P_d/3 \times 10^{-2}$$

Thus the same configuration on the analog can be employed except that now the flow W_o and ΔW must be expressed in terms of P_{do} and ΔP_d.

The plot on the following page gives the results from the analog computer runs. It gives the steady

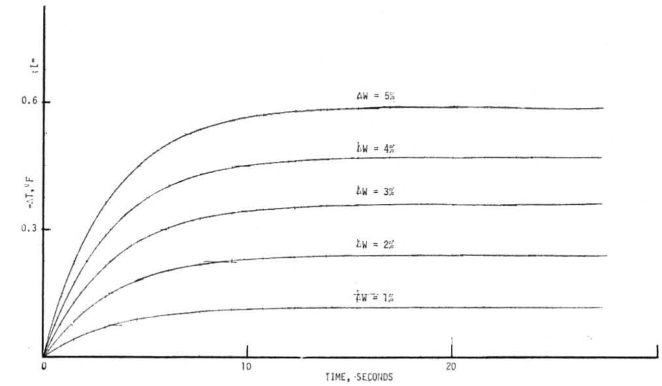


Figure 3
ANALOG SOLUTION OF PERTURBATION EQUATION

temperature T_o, plus the incremental change ΔT for five values of ΔW.

RESULTS

THE RESULTS OF THE ANALOG computer runs for ΔT for the five increments in ΔW agree with the digital computer runs. The data from the digital computer runs are tabulated in Table 4. The first column gives the percentage change in ΔW for each run. The second column gives the percent change in T resulting from the increment ΔW. The third column gives the predicted error in the perturbation equation found by evaluating the small term $\Delta W\ \Delta T/k_2$ which was dropped from the equation. The last column gives the measured error which was found by computing ΔT as the difference between the nominal value T_o and T which was calculated by perturbing the original equation and getting the result to eight significant figures, and comparing

TABLE 4

Comparison of Errors

		$\frac{\Delta T}{T\ \text{max.}}$	E pred.	E meas.
$\frac{\Delta W}{W\ \text{max.}} =$	1%	.0528%	.125%	.429%
	2%	.1056%	.25%	.818%
	3%	.1585%	.375%	1.214%
	4%	.2113%	.50%	1.616%
	5%	.2641%	.625%	2.008%

this with the value of ΔT computed by the perturbation equation.

It can be seen that the measured error, although somewhat larger than predicted, is reasonably small for small perturbations. It can be con-

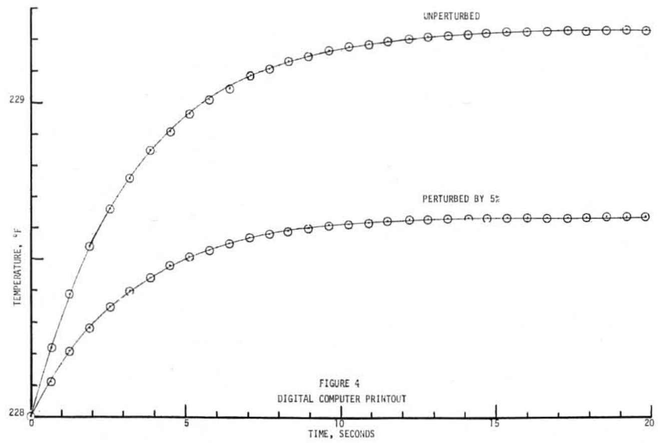


Figure 4
DIGITAL COMPUTER PRINTOUT

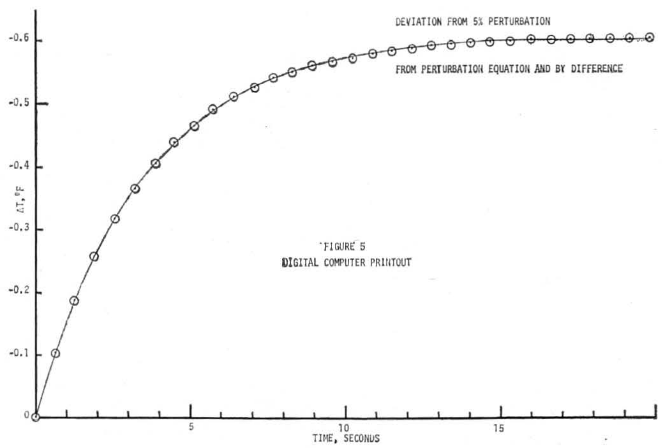


Figure 5
DIGITAL COMPUTER PRINTOUT

cluded then that using the perturbation equation on the analog computer will give much better results than perturbing the original equation and trying to take the difference of nearly equal numbers.

The problem with using conventional techniques, especially on the analog computer, is that the effect on the system response is slight and the isolating of the change by subtraction of nearly equal numbers can produce serious errors. By solving the perturbation equation directly, we can achieve a result which is scaled to the full range of the computer minimizing the error. □

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TABLE 2

Pot#	Parameter	Value	Time Scaled
1	1/2.5	.4	.4
2	$T_i/250$	.8	.8
3	1/2.5	.4	.4
4	$K_i/2.5$	.0704	.704
5	$2W_o/5.K_2$	.05	.50
6	W_o/K_2	.125	1.25
7	K_i	.176	1.76
8	$100\ W/K_2$	.125	1.25
9	$T_s/250$	.9999	.9999

TABLE 3

	CASE 2	
1	1/2.5	SAME AS ABOVE
2	$T_i/250$	
3	1/2.5	
4	$K_i/2.5$	
5	$(2/5)\ ^{o}\left(\sqrt{P_{do}/R_V}\right)/K_2$	
6	$\sqrt{P_{do}/R_V}\ /\ K_2$	
7	K_i	
8	$10\ \Delta P_d/R_V W_o K_2 = 10\left(\frac{\Delta P_d}{K_2\sqrt{P_{do}R_V}}\right)$	
9	$T_s/250$	