ChE class and home problems

The object of this column is to enhance our readers' collection of interesting and novel problems in Chemical Engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class or in a new light or that can be assigned as a novel home problem are requested as well as those that are more traditional in nature that elucidate difficult concepts. Please submit them to Professor H. Scott Fogler, ChE Department, University of Michigan, Ann Arbor, MI 48109.

SOLUTION: PRAIRIE DOG PROBLEM

Editor's Note: Professor Kabel presented the "Prairie Dog Problem" to our readers in the Spring 1980 (Volume XIV, No. 2) issue of Chemical Engineering Education. *The following is his solution to that problem. It is followed by Professor Kabel's new, but similar, "Prairie Dog Appendix" problem statement, for which we invite student solutions.*

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The key point is the Bernoulli effect* and the solution given here pertains to that effect. Of course, students often pursue other mechanisms which lead to good class discussions.

Take the total path length

$2 + 15 + 3 = 20$ m

Assume path cross-sectional area is uniform

1×10^{-2} m^2

Assume laminar flow in burrow and use the Hagen-Poiseuille equation to obtain the flow resulting for a given pressure drop.

Use the Bernoulli equation to relate pressure drop to velocity difference. Bird, Stewart, & Lightfoot (p. 212-213) gives Eq. 7.3-2 as Bernoulli equation with friction included. (This equation is also given as Eq. 2.5-5, p. 30, of Himmelblau and Bischoff.)

*See Vogel, S., and W. L. Bretz, Science 175, 210-211 (1972), the original reference on this and other similar situations, or *Chem. & Eng. News*, May 1, 1972, where I first came across the idea.

$$\Delta \frac{1}{2}\frac{\langle \bar{v}^3\rangle}{\langle \bar{v}\rangle} + \Delta\hat{\phi} + \int_{p_1}^{p_2} \frac{1}{\rho}\,dp + \hat{W} + \hat{E}_v = 0$$

Neglecting work $\hat{W}$, friction $\hat{E}_v$, and potential energy change $\Delta\hat{\phi}$, we get

$$\Delta \frac{1}{2}\frac{\langle \bar{v}^3\rangle}{\langle \bar{v}\rangle} + \int_{p_1}^{p_2} \frac{1}{\rho}\,dp = 0$$

For flat velocity profiles $\langle \bar{v}^3\rangle/\langle \bar{v}\rangle \cong \langle \bar{v}\rangle^2$. Note air flow is characteristically turbulent in open spaces, but the velocity is not really uniform in the region of interest.

Also, for ordinary velocities and pressure, air is incompressible (i.e. ρ = constant). Hence,

$$[\Delta\langle \bar{v}\rangle^2]/2 + (p_2 - p_1)/\rho = 0$$

$$[\langle \bar{v}_2\rangle^2 - \langle \bar{v}_1\rangle^2]/2 + (p_2 - p_1)/\rho = 0$$

Take a very light wind of 0.2 m/sec (= 0.45 mile/hr) at the high port and one-half that at the low port (0.1 m/sec). This selection of conditions is somewhat arbitrary but the magnitude is low enough that such winds can be expected at least on a day-night basis. The factor of 2 is supported by Bird, *et. al.*, p. 136-7, in an example on ideal flow around a cylinder $v^2 = 4\,v_\infty^2 \sin^2\theta$. If the prairie dog mound were a cylinder, the top would be at $\theta = \pi/2$, hence $\sin\theta = 1$ and $v = 2\,v_\infty$. Thus the velocity at the top is twice the approach velocity in this case.

Density of air = 1.2928 g$\cdot$l^{-1} = 1.2928 kg$\cdot$m^{-3} at 0°C, 1 atm or 273.2°K, $1(10^5)$ N$\cdot$m^{-2}

$$-(p_2 - p_1) = \frac{\rho(<\bar{v}_2>^2 - <\bar{v}_1>^2)}{2}$$

$$= 1.2928 \text{ kg}\cdot\text{m}^{-3} \frac{0.2^2 - 0.1^2}{2} \text{m}^2\cdot\text{s}^{-2}$$

$$= 0.0194 \text{ kg}\cdot\text{m}^{-1}\cdot\text{s}^{-2}$$

$$1 \text{ newton} = 1 \text{ kg } (1) \text{m}\cdot\text{s}^{-2} \text{ or } 1 \text{ kg}\cdot\text{s}^{-2} = \text{N}\cdot\text{m}^{-1}$$

$$-(p_2 - p_1) = 0.0194 \text{ N}\cdot\text{m}^{-2} = p_1 - p_2$$

$$= 2.8(10^{-6}) \text{ psi} = 1.45(10^{-4}) \text{ mm Hg}$$

As a matter of interest one student calculated static pressure difference between points 1 and 2 to be 0.0267 $\text{N}\cdot\text{m}^{-2}$.

The Hagen-Poiseuille equation (Eq. 2.3-19, Bird, Stewart, & Lightfoot) gives the air flow rate as

$$Q = \frac{\pi(P_o - P_L) R^4}{8\mu L}$$

where

$P_o - P_L = p_1 - p_2$
$L = 20$ m
$\pi R^2 = 1(10^{-2})\text{m}^2$, $R = 0.0564$ m ($= 2.22$ in)
μ_{air}(0°C, latm) $= 0.01716$ cp $= (1.716)(10^{-5})$ $\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1} = (1.716)(10^{-5})$ $\text{N}\cdot\text{s}\cdot\text{m}^{-2}$

$$Q = \frac{\pi(p_1 - p_2)R^4}{8\mu L} = \frac{\pi(0.0194)(0.0564^4)}{8(1.716)(10^{-5})(20)}$$

$$= (2.246)(10^{-4}) \text{ m}^3\cdot\text{s}^{-1}$$

$$\text{Volume of tunnel} = (20)\text{m}(1)(10^{-2})\text{m}^2 = 0.2\text{m}^3$$

$$\text{Air turnover} = \frac{0.2 \text{ m}^3}{(2.246)(10^{-4}) \text{ m}^3\cdot\text{s}^{-1}} = 890 \text{ s}$$

$$= 14.8 \text{ min} = 0.247 \text{ hr}$$

This answer does not check with the paper exactly since the paper specified turnover every 10 minutes. The difference could be in physical properties but the agreement is close enough and the main point is clear.

$$N_{Re} = D<\bar{v}> \rho/\mu$$

$$= \frac{2(0.0564)\text{m}\left[\frac{(2.246)(10^{-4})}{10^{-2}\text{ m}^2} \frac{\text{m}^3}{\text{s}}\right](1.2928)\text{kg}\cdot\text{m}^{-3}}{1.716(10^{-5}) \text{ kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1}}$$

= 191, so laminar flow does exist in the as assumed.

As a matter of interest the velocity in the tunnel is

$$<\bar{v}> \text{ in channel} = \frac{2.246(10^{-4}) \text{ m}^3\cdot\text{s}^{-1}}{10^{-2} \text{ m}^2}$$

$$= 2.246(10^{-2}) \text{ m}\cdot\text{s}^{-1} = 0.05 \text{ miles}\cdot\text{hr}^{-1}$$

about a factor of 10 slower than outside velocity.

It is clear that by building his mound and burrow properly, the prairie dog will have enough air to breathe even when hibernating. □

A PRAIRIE DOG APPENDIX

Our student readers, both graduate and undergraduate, are encouraged to submit their solution to the following problem to Prof. Robert L. Kabel, ChE Dept., Pennsylvania State University, University Park, PA 16802, before Dec. 31, 1980 (please designate your student status on your entry). A complimentary subscription to CEE will be awarded in each category, to begin immediately or, if preferred, after graduation, for the first correct solution submitted. (Penn State students are not eligible.) We will publish Prof. Kabel's solution in a subsequent issue.

The following numerical example illustrates the effects discussed in the Prairie Dog Problem.

An open vertical tube, 2 m long and 0.01 m in diameter, is to be used as a wind gauge. In a particular experiment the base of the tube was located 0.5 m above ground and an average vertical velocity through the tube was observed to be 1 ms^{-1}. Find the horizontal velocity of the wind at 3 m above the ground. A numerical answer is required.

Other pieces of information which may be helpful are the air density, $\rho = 1.2 \text{ kg m}^{-3}$, and viscosity, $\mu = 1.8(10^{-5}\text{kg m}^{-1} \text{ s}^{-1})$. Also there is a well-known correlation for the horizontal wind velocity, U, as a function of height above the earth's surface, z.

$$\frac{U(z)}{U_*} = \frac{1}{k} \ln \frac{z}{z_0}$$

In this correlation, k is the von Kármán constant and is usually taken equal to 0.4. The roughness height, z_0, for this particular location is 0.04 m. The friction velocity, U_*, is a constant related to the shear stress at the surface which in turn depends on the wind speed.