ChE classroom

ON THE TENSORIAL NATURE OF FLUXES IN CONTINUOUS MEDIA

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IT IS A CURIOUS CIRCUMSTANCE that while courses in fluid mechanics or transport phenomena invest special care to establish the tensorial integrity of the stress system in a fluid, quite often the status of a vector is nonchalantly offered to the mass and energy fluxes. The usual scenario begins with the instructor identifying a point in the fluid continuum, isolating a direction which orients an infinitesimal area dA normal to it and asking for the traction vector representing the net surface force per unit area of dA. The infinite multiplicity of directions at a point leads to the customary despair about its implications to the characterization of surface forces until their redemption through the gift of a second order tensor by the joint effects of the momentum principle and the continuum postulates.

Indeed the foregoing exercise is a healthy one, for the simplicity of description of fluid stress by a second order tensor is not one to be taken for granted. The motivation for this article is the extension of the same considerations to the mass and energy fluxes in a fluid: their claims to being *vectors* is a matter to be established by argument. More generally, the primitive instruments of transport phenomena are orientation-dependent variables, which are an extraordinary encumbrance to mathematical treatment. The central issue here is the replacement of such primitive variables by substitutes that have *no* orientation-dependence for the price of a unit increase in tensorial order. This is a considerable bargain, in the realization of which the conservation laws are intimately involved. Of course, it is not that the dependence of the said primitive variables on orientation has been entirely eliminated but rather that it has been reduced to one of homogeneous linearity. The entire procedure as applied to the traction vector has been called Cauchy's fundamental theorem.†

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If it is admitted that the velocity v is a vector, then it would instantly follow that the mass flux ρv is a vector and no further 'proof' would be called for. On the other hand, an independent definition of point velocity in a deforming continuum does not build on a limiting procedure of a collapsing mass, a requirement basic to proper definitions of "point" quantities. The alternative is then to seek a suitable definition of something related to velocity (such as mass flux) from which the velocity may be obtained as a derived quantity. The choice falls somewhat naturally on mass flux which, as mass flow rate per unit area, could depend on the orientation of the area. This complex situation is alleviated by the fact that the mass flux turns out to be a vector. Similar considerations also hold for the energy flux. Before demonstrating these results for mass and energy fluxes, it is instructive to review the derivation of Cauchy's fundamental theorem for stress, which is related to the modeling of mechanical interactions within a material continuum.

†See for example, "The Elements of Continuum Mechanics," by C. Truesdell, Springer Verlag, New York, Inc., 1965.

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The authors recall the many pleasant years of their interaction during their early academic careers at the I. I. T., Kanpur, where the academic environment was extraordinary. The present article grew out of an evening's discussion many years ago. **(R)**

THE STRESS TENSOR

At a point $\mathbf{x} = [x_i]$ in a continuous body consider a small area dA whose orientation is determined by its unit normal $\mathbf{n} = [n_i]$. Then, the mechanical action of one part of the body on its neighborhood across dA can be represented by a force density $\mathbf{t}$ such that $\mathbf{t}$ dA gives the force exerted on the area dA. Because of the nature of force $\mathbf{t}$ is a vector, called the traction vector, which has the dimensions of stress-force per unit area. Besides being a function of the position $\mathbf{x}$ and time t, the traction vector $\mathbf{t} = \mathbf{t}(\mathbf{x},t;\mathbf{n})$ is also a function of the orientation $\mathbf{n}$ of the area dA. As a result, the traction vector is not suitable for purposes of analysis, even though it is the primitive variable of interest. Cauchy's fundamental theorem guarantees the existence of a second order tensor $\mathbf{T}(\mathbf{x},t) = [T_{ij}]$, called the stress tensor, such that

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \mathbf{n}\cdot\mathbf{T}(\mathbf{x},t)$$

or (1)

$$t_i(\mathbf{x},t;\mathbf{n}) = n_j T_{ji}(\mathbf{x},t) \quad \text{, summation over j}$$

The proof of this result is in two steps. First we show that $\mathbf{t}(\mathbf{x},t;-\mathbf{n}) = -\mathbf{t}(\mathbf{x},t;\mathbf{n})$. This follows from an application of Euler's first law of motion (the rate of change in linear momentum of a body is equal to the total force applied to the body) to a thin cylindrical disk of thickness dl and cross-sectional area dA, as shown in Figure 1, which results in the equation of motion

$$\rho(dl\, dA)\mathbf{a} = dA[\mathbf{t}(\mathbf{x},t;\mathbf{n}) + \mathbf{t}(\mathbf{x},t;-\mathbf{n})] + dl\int_c \mathbf{t}(\mathbf{x},t;s)\,ds + \rho(dl\, dA)\mathbf{f}(\mathbf{x},t)$$

where $\mathbf{a}$ is the acceleration of the material disk, ρ is the density of the material, the line integral along the contour c gives the contribution to the force due to the stress vector $\mathbf{t}(\mathbf{x},t;s)$ acting on the

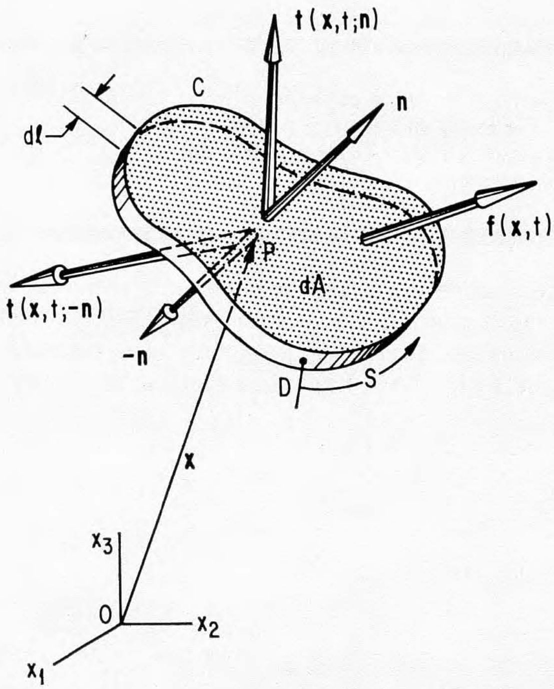

FIGURE 1. Traction vectors acting on a thin disk.

cylindrical surface of the disk, and the last term is the contribution due to the body force. Taking the limit $dl \rightarrow 0$ in Eq. (1) and dividing the resulting equation by dA yields

$$\mathbf{t}(\mathbf{x},t;-\mathbf{n}) = -\mathbf{t}(\mathbf{x},t;\mathbf{n}) \qquad (2)$$

Next we consider the motion of the small material tetrahedron, PABC, at the point $\mathbf{x}$, as shown in Fig. 2, whose edges are along the orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and where face ABC has an area dA and normal $\mathbf{n} = [n_i]$. The areas of the faces with normals $-\mathbf{e}_r$ are then given by $dA_r = n_r dA$, i = 1,2,3. Let the distance of P from the face ABC be ds. Then, an application of Euler's first law of motion to the material tetrahedron results in the equation

$$\rho\left(\frac{1}{3}dsdA\right)\mathbf{a} = dA\left[\mathbf{t}(\mathbf{x},t;\mathbf{n}) + \sum_{r=1}^{3} n_r \mathbf{t}(\mathbf{x},t;-\mathbf{e}_r)\right] \qquad (3)$$

$$+ \rho\left(\frac{1}{3}dsdA\right)\mathbf{f}(\mathbf{x},t)$$

Division of Eq. (3) by dA, followed by taking the limit as $ds \rightarrow 0$, and the use of Eq. (1) then gives

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \sum_{r=1}^{3} n_r \mathbf{t}(\mathbf{x},t;\mathbf{e}_r). \qquad (4)$$

This equation shows that if the traction vector $\mathbf{t}$ is known at a point on three mutually orthogonal planes (here $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$), then it can be determined on any plane $\mathbf{n}$ at that point. We can reduce this result further: Let the components of $\mathbf{t}(\mathbf{x},t;\mathbf{e}_r)$ along the axes $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be given by

$$\mathbf{t}(\mathbf{x},t;\mathbf{e}_r) = \sum_{s=1}^{3} T_{rs}\,\mathbf{e}_s.$$

Then Eq. (4) becomes

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \sum_{r=1}^{3}\sum_{s=1}^{3} n_r T_{rs}\,\mathbf{e}_s$$

$$= \mathbf{n}\cdot\sum_{r=1}^{3}\sum_{s=1}^{3} T_{rs}\,\mathbf{e}_r\,\mathbf{e}_s$$

Thus

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \mathbf{n}\cdot\mathbf{T}(\mathbf{x},t), \qquad (5)$$

The same procedure could be used for mass and energy fluxes, both of which arise from orientation dependent scalars.

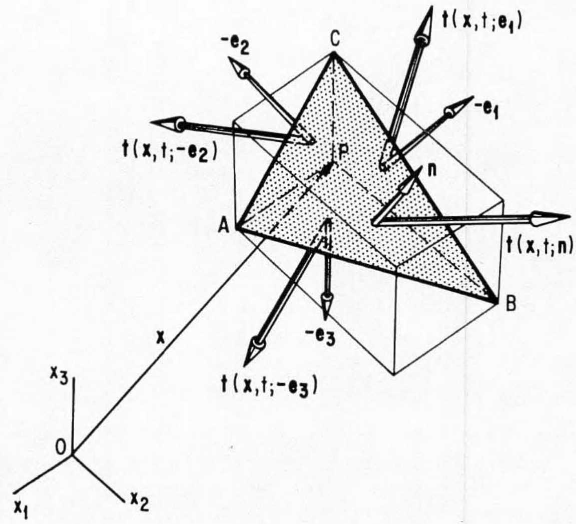

FIGURE 2. Traction vectors acting on a infinitesimal tetrahedron.

where $\mathbf{T}$ is the dyadic

$$\mathbf{T} = \sum_{r=1}^{3}\sum_{s=1}^{3} T_{rs}\,\mathbf{e}_r\,\mathbf{e}_s.$$

If Cartesian tensors are used, with summation implied over a repeated index, then this equation can be written alternatively as

$$t_i(\mathbf{x},t;\mathbf{n}) = n_r\,T_{ri}(\mathbf{x},t)$$

Let us summarize what we have so far. We started with a primitive variable of interest—the traction vector which is orientation dependent. By using a physical principle, here Euler's first law of motion, we were able to establish the existence of a second order tensor—the stress dyadic $\mathbf{T}(\mathbf{x},t)$, which is a point function and which factors out the dependence of $\mathbf{t}$ on $\mathbf{n}$.

The same procedure could be used for mass and energy fluxes, both of which arise from orientation dependent scalars. Application of physical laws—here conservation of mass and the first law of thermodynamics, respectively, would result in suitable point functions that are vectors which factor out the dependence of the primitive scalars on the orientation $\mathbf{n}$. In each case the desired result would be obtained because, in the equation which results as a consequence of the application of the physical principle, the volume dependent terms vanish faster than the surface dependent terms in the limit when one of the dimensions of the body approaches zero—just as in the derivation of Eqs. (2) and (5). Instead of going through this exercise individually for each case, it is useful to formalize this property in the form of a theorem that is discussed in the following section. Since the result applies to tensors of all orders, it is more descriptive to use Cartesian tensors rather than Gibbsian dyadics and polyadics. However, each result will also be presented in its Gibbsian polyadic form.

A THEOREM FOR TENSOR FIELDS

Consider a cartesian tensor *field* of order p (the case of p = 0 represents a scalar and that for p = 1 represents a vector), $\mathbf{b} = [b_{i_1 i_2 \cdots i_p}]$, whose components at time t depend not only on the spatial coordinates $\mathbf{x} = [x_i]$ of the point, say P at which they are considered but also on a specified orientation $\mathbf{n} = [n_i]$ at P. The foregoing dependence on spatial coordinates and orientation is presumed to be *continuous* and we write

$$b_{i_1 i_2 \cdots i_p} = b_{i_1 i_2 \cdots i_p}(\mathbf{x},t;\mathbf{n})$$

or

$$\mathbf{b} = \mathbf{b}(\mathbf{x},t;\mathbf{n})$$

Before we state the theorem of interest, some preparations are essential. At *any point* P in the continuum we identify a volume V bounded by an area A enclosing P. The surface integral

$$\frac{1}{A}\iint_A b_{i_1 i_2 \cdots i_p}(\mathbf{x},t;\mathbf{n})\,dA$$

or

$$\frac{1}{A}\iint_A \mathbf{b}(\mathbf{x},t;\mathbf{n})\,dA$$

where **n** is everywhere *normal* to area *A* and pointing *outwards,* is defined for every volume enclosing P and in fact, through the mean value theorem, equals the value of the integrand somewhere on the area A. It is now possible to take the limit of this surface mean by contracting the volume to zero around P.

The theorem concerned states that there *exists* a tensor of order p + 1, denoted by $\mathbf{B} = [B_{j i_1 i_2 \cdots i_p}(\mathbf{x},t)]$ such that

$$b_{i_1 i_2 \cdots i_p}(\mathbf{x},t;n) = n_j B_{j i_1 i_2 \cdots i_p}(\mathbf{x},t), \text{ summation on } j, \qquad (6)$$

which is $\mathbf{b}(\mathbf{x},t;\mathbf{n}) = \mathbf{n}\cdot\mathbf{B}(\mathbf{x},t)$ in Gibbsian notation, *if and only if*

$$\lim_{V\to 0}\frac{1}{A}\iint_A b_{i_1 i_2 \cdots i_p}(\mathbf{x},t;\mathbf{n})\,dA = 0$$

i.e.

$$\lim_{V\to 0}\frac{1}{A}\iint_A \mathbf{b}(\mathbf{x},t;\mathbf{n})\,dA = 0 \qquad (7)$$

Note in particular that the new tensor $\mathbf{B}(\mathbf{x},t) = [B_{j i_1 i_2 \cdots i_p}]$ is not a function of the vector **n**.

The proof of this theorem is straightforward.

Thus the conservation principles are themselves responsible for "fluxes" being vectors or tensors, a result of tremendous significance in the investigation of the mechanics of continua.

To prove the 'only if' part we assume that Eq. (6) is true. Thus we establish Eq. (7) by

$$\lim_{V\to 0}\frac{1}{A}\iint_A b_{i_1 i_2 \cdots i_p}(\mathbf{x},t;\mathbf{n})\,dA = \lim_{V\to 0}\frac{1}{A}\iiint_V \frac{\partial}{\partial x_j}[B_{j i_1 i_2 \cdots i_p}]\,dV = 0$$

where we have used (i) the divergence theorem, (ii) the summation convention on index j and (iii) the assumption that the divergence $(\partial B_j\ldots/\partial x_j)$ is continuous and bounded. For the converse, we start from Eq. (7). By taking V to be a right cylinder of infinitesimal cross-section dA with its normal along **n**, as shown in Figure 1, and contracting the cylinder along its axis to a section passing through **x**, we conclude that

$$b\ldots(\mathbf{x},t;-\mathbf{n}) = -b\ldots(\mathbf{x},t;\mathbf{n})$$

or

$$\mathbf{b}(\mathbf{x},t;-\mathbf{n}) = -\mathbf{b}(\mathbf{x},t;\mathbf{n}) \qquad (8)$$

where for simplicity the indices have been replaced by dots. Next, at any point **x** (position vector referred to some origin) and orientation **n**, we apply the result of Eq. (7) to a sequence of tetrahedra such that each of them has a vertex at **x** and the face opposite is *normal* to **n**, which points *out* of the tetrahedron (see Figure 2). The other faces are formed such that the edges passing through **x** lie along three mutually perpendicular lines chosen to be a set of cartesian coordinate directions. The normals pointing out of these faces are the unit vectors $-\mathbf{e}_1$, $-\mathbf{e}_2$ and $-\mathbf{e}_3$ respectively. The application of Eq. (7) to this collapsing sequence together with Eq. (8) gives

$$\mathrm{b}...(\mathbf{x},t;\mathbf{n}) = \mathbf{n}_j\, \mathrm{b}...(\mathbf{x},t;\mathbf{e}_j), \text{ summation over j,} \quad (9)$$

or

$$\mathbf{b}(\mathbf{x},t;\mathbf{n}) = \sum_{j=1}^{3} n_j\, \mathbf{b}(\mathbf{x},t;\mathbf{e}_j)$$

$$= \mathbf{n} \cdot \sum_{j=1}^{3} \mathbf{e}_j \mathbf{b}(\mathbf{x},t;\mathbf{e}_j)$$

Since b... $(\mathbf{x},t;\mathbf{n})$ and n_j are tensors of orders p and 1, respectively, it follows from the quotient law that $\sum_{j=1}^{3} \mathbf{e}_j \mathbf{b}(\mathbf{x},t;\mathbf{e}_j)$ is a tensor of order (p+1). Thus the theorem is proved with

$$B_j...(\mathbf{x},t) = \mathrm{b}...(\mathbf{x},t;\mathbf{e}_j)$$

or

$$\mathbf{B}(\mathbf{x},t) = \sum_{j=1}^{3} \mathbf{e}_j\, \mathbf{b}(\mathbf{x},t;\mathbf{e}_j)$$

MASS FLUX IS A VECTOR

At any point **x** in the deforming continuum, given a direction **n** it is possible to define a mass flow rate per unit area (a continuum postulate!) across an area normal to **n**. Denoting this by $m(\mathbf{x},t;\mathbf{n})$, we obtain the mass flow rate through dA by $m(\mathbf{x},t;\mathbf{n})dA$. Conservation of mass for a volume V bounded by A gives

$$\frac{d}{dt} \iiint_V \rho dV + \iint_A m(\mathbf{x},t;\mathbf{n})dA = 0 \quad (10)$$

where ρ is the fluid density. If Eq. (10) is divided by A and we let $V \rightarrow 0$, the continuum postulate that $\partial\rho/\partial t$ be continuous on V and A leads to the result

$$\lim_{V\rightarrow 0} \frac{1}{A} \iint_A m(\mathbf{x},t;\mathbf{n})dA = 0$$

which is precisely the condition given by Eq. (7) so that the theorem just discussed leads to the existence of a *mass flux vector* $\mathbf{M}(\mathbf{x},t)$ such that

$$m(\mathbf{x},t;\mathbf{n}) = n_j M_j(\mathbf{x},t)$$

or

$$m(\mathbf{x},t;\mathbf{n}) = \mathbf{n} \cdot \mathbf{M}(\mathbf{x},t)$$

Thus the mass flow rate through area $d\mathbf{A} \equiv \mathbf{n}dA$ is given by $d\mathbf{A} \cdot \mathbf{M}$. The point velocity vector is now readily defined by $\mathbf{v} = (1/\rho)\mathbf{M}$. In an n-component mixture, the mass flux of the k^{th} component of mass fraction ω_k is defined $\mathbf{M}_{(k)} = \mathbf{M}\omega_k$, from which its velocity is defined by $\mathbf{v}_{(k)} = (1/\rho_k)\mathbf{M}_{(k)}$.

STRESS IS A SECOND ORDER TENSOR

Although the stress tensor has been covered earlier, the repetition here will serve this tutorial well by reinforcing the result as a fallout from the generalized theorem above.

A momentum balance for the volume of fluid V bounded by A must account for the traffic of momentum across A because of mass flow, body forces throughout V and surface forces along the entire surface A. The traction (force per unit area) on an area dA, located at **x** on A, with normal **n** (directed out of V) is denoted by $[t_i(\mathbf{x},t;\mathbf{n}]$. For the i^{th} component one has

$$\frac{d}{dt} \iiint_V \rho v_i dV + \iint_A v_i M_j dA_j = \iiint_V \rho f_i dV + \iint_A t_i(\mathbf{x},t;\mathbf{n})dA \quad (11)$$

where $[f_i]$ represent components of the body force. Dividing Eq. (11) by A and letting $V\rightarrow 0$, the volume integrals are readily seen to vanish. Since we have shown that **M** is a vector, $[M_j v_j]$ transforms as a vector for each fixed i and the theorem is applicable to the second term on the left hand side of Eq. (11). Thus we obtain

$$\lim_{V\rightarrow 0} \frac{1}{A} \iint_A t_i(\mathbf{x},t;\mathbf{n})d\mathbf{A} = 0$$

so that the theorem is again applicable, and yields second order tensor $[T_{ji}(\mathbf{x},t)]$ such that

$$t_i(\mathbf{x},t;n) = n_j T_{ji}(\mathbf{x},t)$$

or

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \mathbf{n} \cdot \mathbf{T}(\mathbf{x},t)$$

The surface force on an area $d\mathbf{A} \equiv \mathbf{n}dA$ has for its i^{th} component $dA_j T_{ji}$.

ENERGY FLUX IS A VECTOR

By energy flux here we mean that which occurs by molecular conduction.† Of course energy is transported by fluid motion but since this is associated with the mass flux vector the resulting energy flux is already known to be a vector. Thus our con-

†Radiative transport through weakly absorbing media would require a slightly more elaborate treatment.

cern here belongs to the energy transported by conduction per unit area across an area located at $\mathbf{x}$ and normal to $\mathbf{n}$, denoted by $q(\mathbf{x},t;\mathbf{n})$. The first law of thermodynamics applied to volume V bounded by A, gives

$$\frac{d}{dt}\iiint_V [\hat{u} + \frac{1}{2}v^2]\,dV + \iint_A [\hat{h} + \frac{1}{2}v^2]M_j dA_j$$
$$= \iint_A dA_j T_{ji} v_i + \iiint_V \rho f_i v_i dV - \iint_A q(\mathbf{x},t;\mathbf{n})\,dA \quad (12)$$

The interpretations of the various terms in Eq. (12) are available in any standard textbook. In view of the pattern already set before we are able to conclude that

$$\lim_{V\to 0} \frac{1}{A} \iint_A q(\mathbf{x},t;\mathbf{n})\,dA = 0$$

so that there exists an energy flux vector $[Q_j(\mathbf{x},t)]$ such that

$$q(\mathbf{x},t;\mathbf{n}) = n_j Q_j(\mathbf{x},t)$$

or

$$q(\mathbf{x},t;\mathbf{n}) = \mathbf{n} \cdot \mathbf{Q}(\mathbf{x},t)$$

Thus the conservation principles are themselves responsible for "fluxes" being vectors or tensors, a result of tremendous significance in the investigation of the mechanics of continua. It is a fact that deserves mention in courses on fluid mechanics and transport phenomena.

ACKNOWLEDGMENT

Personal discussions have borne out that the issues raised in this paper are a routine matter to many. The authors would like to specially acknowledge Professors L. E. Scriven at the University of Minnesota and Stephen Whitaker at the University of California, Davis. In particular, Professor Whitaker was kind enough to provide evidence that the contents of this paper are not common knowledge and represent useful information. □

COMPETITIVE ENVIRONMENT

Continued from page 81.

for product "C", Team II could have gained the largest profits.

Upon the completion of 10 operating periods, a year by year summary of the performance of all teams is made available to the participants. Each team is asked to prepare a report to analyze their performance and identify the important decisions and actions that led to their relative position in the competition. Although the simple "return on investment" is suggested as a possible economic evaluator of performance, the teams are at liberty to select alternate evaluators; e.g., "internal rate of return" (the interest rate that makes the cumulative discounted cash flow equal to zero) or "borrowing power". As would be expected, those groups that make significant profits in the early periods of the simulation tend to base comparisons on the "internal rate of return". The use of these alternate measures of performance can cause reversals in the relative positions established by comparisons based on "net profit" or the simple "return on investment" criteria.

CONCLUSION

Student response to this project has been enthusiastic. The immediate consequence of their decisions provides a sense of realism for the interplay between technical, marketing, and economic factors. The computer simulator, the package of memoranda, and instruction manual are available at a total cost of $275.00 from Engineering Educational Materials, 805 Baylor Drive, Newark, Delaware 19711. □

ACKNOWLEDGMENTS

We would like to thank all those who have helped us in the development of this educational tool. Mr. R. N. Pratt, Engineering Computation Specialist in the College of Engineering at the University of Delaware, has been most helpful in setting up and modifying the computer program for the simulation. His efforts in running the simulation are also greatly appreciated. Student feedback from experiences in playing the game was essential to proper game development and their input is gratefully acknowledged.

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