ChE classroom

A PHYSICAL INTERPRETATION FOR THE GAMMA DISTRIBUTION

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STATISTICAL DISTRIBUTIONS have been used extensively in chemical engineering to characterize distributions of residence times in flow systems such as reactors, packed beds, crystallizers and other process equipment. The two extreme situations of plug flow and complete backmixing are represented by the uniform distribution and the exponential distribution respectively [1]. Non-ideal flow situations between these extreme cases have been accounted for by popular models such as the dispersion model, the tanks in series model and combinations of plug flow and backmixed models with dead zones.

Gamma distributions have been used in the literature for the characterization of distributions such as the residence time distribution (RTD) and crystal size distribution (CSD). This distribution can be used for random variables with non-negative values. The gamma distribution has been either derived from fundamentals, as in the case of the hopping model by Rathor *et al* [2] for merging and splitting stream, or used empirically for describing the distribution of a random variable such as growth rate of the crystals by Larson *et al*, [3]. This paper examines a physical interpretation for a gamma residence time distribution through a statistical-mathematical analysis.

Consider N identical tanks in series with no inter-

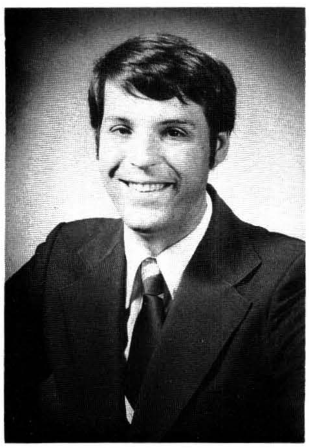

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TABLE 1
Statistical Background Material

Probability density function $f_X(x)$ is defined such that if x is a random variable, the probability that $x \epsilon (x, x + dx) = f_X(x)dx$.

Expected value of x^j is defined as

$$E\left(x^j\right) = \int_{-\infty}^{\infty} x^j f_X(x)dx = M_X(j) \qquad (1.1)$$

Note that the $E[x^j]$ = is also the j^{th} moment of the probability distribution about the origin, $M_X(j)$.

For a non-negative random variable

$$E\left(x^j\right) = \int_{0}^{\infty} x^j f_X(x)dx = M_X(j) \qquad (1.2)$$

where $M_X(j) = j^{th}$ moment about the origin of $f_X(x)$

Similarly, the expected value of a function g(x) can be defined as

$$E[g(x)] = \int_{0}^{\infty} g(x) f_X(x)dx \qquad (1.3)$$

Consider the special case when

$$g(x) = \exp(-sx) \qquad (1.4)$$

thus

$$E[\exp(-sx)] = \int_{0}^{\infty} \exp(-sx)\, f_X(x)dx = \psi_X(s) \qquad (1.5)$$

$\psi_X(s)$ is the Laplace Transform of the probability density function, $f_X(x)$ and s is the Laplace variable for the random variable x.

mixing between the tanks. Let τ represent the mean residence time of the fluid element in each tank.

The total residence time of any fluid element in the N tank system t is

$$t = t_1 + t_2 + t_3 + t_4 \ldots + t_i \ldots + t_N \tag{1}$$

where t_i represents the residence time of the fluid element in the i^{th} tank. Since each tank is assumed to be well mixed, t_i is a random variable distributed exponentially having a distribution function [1]

$$f_T(t_i) = \frac{1}{\tau} \exp\left(\frac{t_i}{\tau}\right) \tag{2}$$

Let s represent the Laplace variable for t_i. From Eq. (1)

$$\exp(-st) = \exp(-st_1) \exp(-st_2) \ldots \exp(-st_N) \tag{3}$$

Taking the expected values of both sides of Eq. (3) with the assumption that t_i's are random independent variables

$$E[\exp(-st)] = E[\exp(-st_1)]\, E[\exp(-st_2)] \ldots \ldots E[\exp(-st_N)] \tag{4}$$

Since each tank is well-mixed and the residence time distribution in the tank is represented by

$$f_T(t_i) = \frac{1}{\tau} \exp\left(\frac{t_i}{\tau}\right) \tag{5}$$

$$E[\exp(-st_i)] = \int_0^\infty \exp(-st_i) \frac{1}{\tau} \exp\left(\frac{t}{\tau}\right) dt_i \tag{6}$$

$$= \frac{1}{s\tau + 1} \tag{7}$$

Non-ideal flow situations between these extreme cases have been accounted for by popular models such as the dispersion model, the tanks in series model and combinations of plug flow and back-mixed models with dead zones.

or

$$E[\exp(-st)] = \left(\frac{1}{s\tau + 1}\right)^N \tag{8}$$

But from the definitions presented in Table 1, E[exp(-s t)] is the Laplace transform of the function $f_T(t)$ with respect to t, $\psi_T(s)$ [4].

Taking the inverse Laplace transform of Eq. (8),

$$f_T(t) = \frac{1}{\tau^N \,\Gamma(N)} t^{N-1} \exp\left(\frac{t}{\tau}\right) \tag{9}$$

Comparing Eq. (9) with the standard equation for a gamma distribution [5]

$$f_X(x) = \frac{1}{\beta^\alpha \,\Gamma(\alpha)} x^{\alpha-1} \exp\left(\frac{x}{\beta}\right) \tag{10}$$

it is evident that α corresponds to the number of identical tanks and β corresponds to the residence time of the fluid in each tank.

Table 1 lists some of the useful properties of the gamma distribution. The product $\alpha\beta$ represents the mean value of the random variable, in this case the mean residence time of the fluid elements in the N tank system. The variance of the distribution is $\alpha\beta^2$.

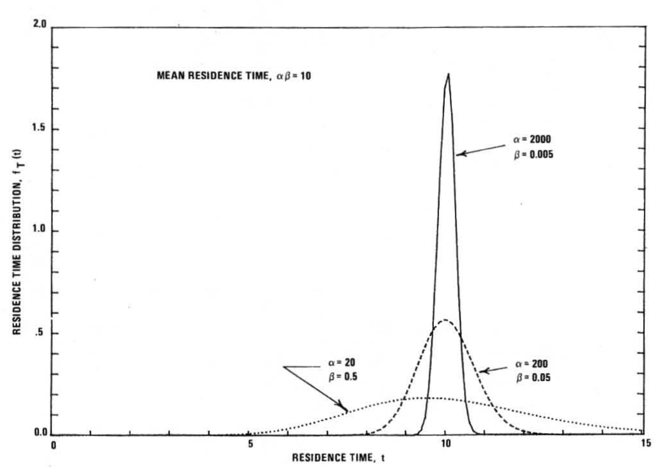

FIGURE 1. Typical plots of the gamma residence time distribution for large values of α.

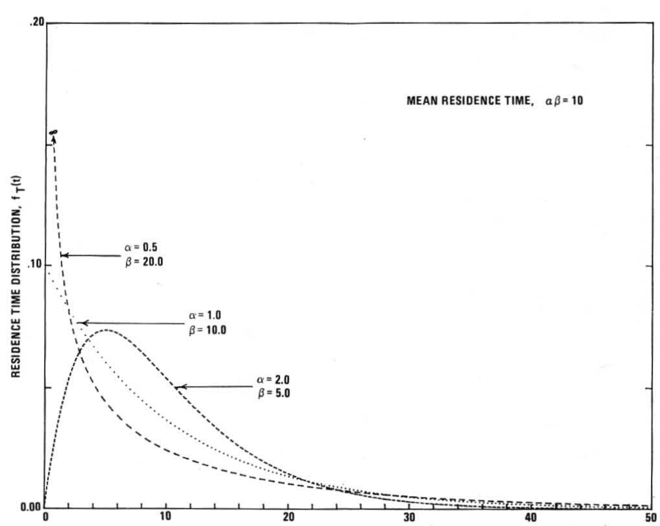

FIGURE 2. Typical plots of the gamma residence time distribution for small values of α.

TABLE 2
Properties of the Gamma Residence Time Distribution

Form: $f_T(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} t^{\alpha-1} \exp\left(-\frac{t}{\beta}\right) \quad \alpha > 0,\ \beta > 0$

1) $\int_0^\infty f_T(t) = 1$

2) $f_T(t=0) = 0$

3) Mean of the distribution, $\bar{\tau} = \alpha\beta$

4) Variance, $\sigma_T^2 = \alpha\beta^2$

5) $(CV)_T = 1/\sqrt{\alpha}$

6) The j^{th} moment of $f_T(t)$ about the origin:

$$M_T(j) = \frac{\beta^j \Gamma(\alpha + j)}{\Gamma(\alpha)}$$

7) $\psi_T(s) = \frac{1}{(\beta s + 1)^\alpha}$

where s is the Laplace Transform variable for t.

For the special case when $\alpha = 1$, the gamma distribution reduces to an exponential distribution (one well-mixed tank) and β corresponds to the mean residence time in one tank τ.

Figures 1 and 2 depict the shapes of the gamma distribution for different combinations of α and β with the same mean $\alpha\beta$. It can be seen that for a large value of α, the RTD is narrow and corresponds to the plug flow situation illustrating that the plug flow system is nothing but a combination of a large number of small, well-mixed systems. Alternately, the gamma distribution can be interpreted as the sum of independent identical exponential distributions.

NOTATION

$(CV)_T$	coefficient of variation for the RTD
$E[x]$	expected value of the random variable x
$f_X(x)$	probability density function for the random variable x
$g(x)$	moment generating function
$M_X(j)$	j^{th} moment about the origin of the probability density function $f_X(x)$
N	number of identical tanks in series
s	Laplace transform variable for t
t	residence time of fluid element in the system
$<t>$	mean of the residence time distribution

Greek Letters

α	parameter for the gamma residence time distribution
β	parameter for the gamma residence time distribution
$\Gamma(n)$	gamma function
$\Gamma(\alpha,\beta)$	gamma probability density function with parameters α and β
$\psi_x(s)$	Laplace transform of the probability density function $f_X(x)$ with respect to the random variable x
τ	mean residence time

Subscripts

T	residence time distribution

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REVIEW: ChE Thermodynamics

Continued from page 21.

count for thermal energy flow and work done by a spontaneous process: for example, a power plant?

Thermodynamics is an exact mathematical structure based on exact differential forms. The relationship between this structure and the independent variables used to identify the interactions of mass and energy in physical systems and their surroundings requires tight definitions, statements about the systems and the interactions between the system and the surroundings. Professor Daubert has not been very careful with these details in this book.

The solved problems and exercises presented in this book are certainly instructive. The ordering of the material is unconventional but it will probably work in the classroom. Take a careful look at the theoretical development in this book. The application of thermodynamics to a new problem is the test students must pass after completing the thermodynamics course. It is the careful development of these analytical skills that are missing in this book. □