The object of this column is to enhance our readers' collection of interesting and novel problems in chemical engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class, or in a new light, or that can be assigned as a novel home problem, are requested as well as those that are more traditional in nature, which elucidate difficult concepts. Please submit them to Professor H. Scott Fogler, ChE Department, University of Michigan, Ann Arbor, MI 48109.

A PROBLEM WITH COYOTES

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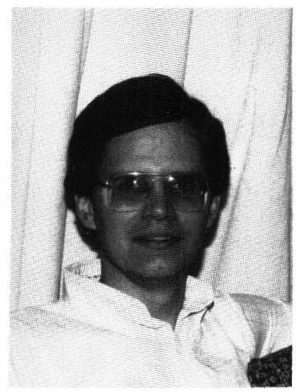
As a student in a graduate reaction engineering course, I was assigned the task of creating and taking a final examination for the course.* In our class discussion of reactor stability we had briefly addressed limit cycle behavior and its representation using phase-plane plots. This was the third instance in a matter of months that I had heard reference made to limit cycle behavior. The topic had also been broached in a departmental seminar and in another class. However, in each case the speaker did not have time to elaborate on this intuitively puzzling phenomenon. Hence, it seemed that a problem involving this stability concept would be interesting to the imaginary student taking my test.

A microbial predator-prey interaction model that I had been exposed to in a biotechnology course provided an attractive starting point, mostly due to its simplicity. However, I chose to apply the microbial model to a mammalian system, with the thought that such a macroscopic system would be easier to visualize. In an ancillary question, I observed that I had applied a simple model to a complex system and called upon the student to critique the model's construction and to propose possibilities for its improvement. The question and its solution follow.



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PROBLEM

While working in Arizona as a petroleum engineer, you are befriended by a sheep rancher who lives down the road. One afternoon the rancher seeks your advice on a problem. Recently his flock has been plagued by coyote attacks. In fact, in recent years so many lambs have been lost to coyotes that his flock is decreasing in size, a situation resulting in significant economic hardship. An acquaintance of his at the FCX has offered to trap and destroy coyotes in his region. The fee, however, is exorbitant. Nevertheless, the rancher is tempted to try the measure in hopes of expanding the sheep population.

You vaguely recall reading about the Lotka-Volterra model of predator-prey interactions during your university days. Leafing through an old book*, you find the following equations for the model

$$\frac{dn_1}{dt} = an_1 - kn_1n_2$$

$$\frac{dn_2}{dt} = -bn_2 + qkn_1n_2$$

*For a discussion of this assignment, see Felder, R. M., "The Generic Quiz: A Device To Stimulate Creativity and Higher-Level Thinking Skills," *Chemical Engineering Education*, 19, 176 (1985).

*Bailey, J. E. and Ollis, D. F., *Biochemical Engineering Fundamentals*, [2nd Ed.] New York: McGraw-Hill Book Co., 1986.

where n_1 = prey population

n_2 = predator population

a,b = specific growth rate constants for prey and predator respectively (time^{-1})

n_1n_2 = product of predator-prey populations; proportional to the frequency of predator-prey encounters

k = proportionality constant; represents both the fraction of predator-prey encounters resulting in death of the prey and the rate of decrease in prey population per kill (time^{-1} coyote^{-1})

q = proportionality constant; represents the amount by which predator population increases per kill (coyote sheep^{-1})

The book also states that the model may be expressed as

$$\left(\frac{y_1}{\exp(y_1)}\right)^b \left(\frac{y_2}{\exp(y_2)}\right)^a = \exp(c)$$

where

$$y_1 = \left(\frac{n_1}{n_{1s}}\right) \qquad y_2 = \left(\frac{n_2}{n_{2s}}\right) \qquad c = \text{integration constant}$$

and the steady state solutions, n_{1s} and n_{2s}, are

$$n_{1s} = \left(\frac{b}{qk}\right) \qquad n_{2s} = \left(\frac{a}{k}\right)$$

1. Derive the second form of the model beginning with the first.

2. The trapper estimates that his operation could

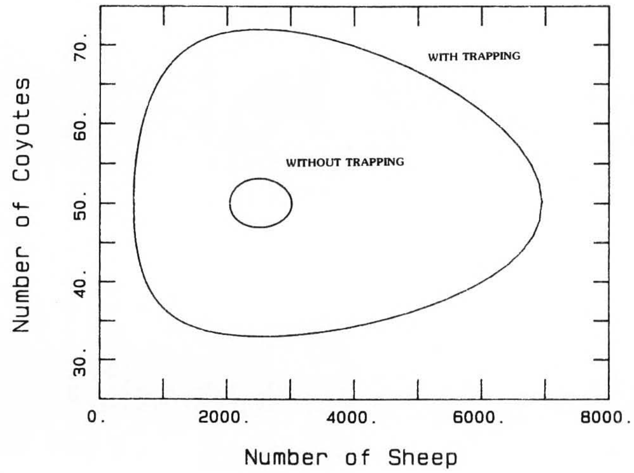

FIGURE 1. Predator/Prey Population Cycles

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provide a 38% reduction in the coyote population. Assuming the following values for model parameters, would you recommend the trapping operation based upon the Lotka-Volterra model? In your analysis consider the time dependence of the sheep population both before and after the proposed trapping operation. Summarize your findings using phase-plane plots.

Data: a = 5×10^{-3} day^{-1}

b = 5×10^{-4} day^{-1}

k = 10^{-4} day^{-1} coyote^{-1}

q = .002 coyote sheep^{-1}

n_1 (initial) = 2350 sheep

n_2 (initial) = 53 coyotes

3. What assumptions have been integral to your analysis which might affect the validity of your results? How might you modify the model to increase its applicability for this situation?

SOLUTION

1. The derivation is easily performed and is briefly outlined by Bailey and Ollis (p. 872).

2. The behavior of the two populations over time may be represented in a phase-plane plot, which could be generated by either of two means: The second form of the model could be solved for y_2 for a selected y_1, or the coupled equations could be solved directly via a numerical technique. The highly nonlinear nature of the second model expression makes determination of its roots via conventional numerical techniques quite difficult. An additional disadvantage to this approach is that one cannot associate a time with a given position on the plot, which might be helpful in an application such as this. Consequently, the coupled equations were solved using a Runge-Kutta routine. The output appears in Table 1 (next page).

The phase-plane plot appears in Figure 1. Shown are the predicted population cycles for the situations with and without the decrease in coyote population. The stable population of 2500 sheep and 50 coyotes is indicated. From the graph and the data one would deduce that the sheep population is currently about halfway through the declining phase of its cycle, which correlates with the rancher's account of dwindling numbers of sheep in recent years. Although the population is cyclic, it is relatively close to the stable population. On the other hand, after the elimination of 20 coyotes, the range of the cycle becomes enormous. If this cycle were followed, the sheep population would soar to over 6500 for a period but then plummet to below 1000 for over three years. To determine whether the trapping operation would lead to a net increase in the average sheep population, one can time-average the data for both situations:

$$\frac{\text{average number of sheep}}{\text{year}} = \frac{1}{T}\int_0^T n_1 \, dt$$

where T = the cycle period

Taking this average using the trapezoid rule yields

average population without trapping = 2500 sheep
average population with trapping = 2500 sheep

Thus, in either case the population oscillates around the same value. Consequently, the rancher would be unwise to pay for trapping the coyotes. Reduction of the coyote population would not increase his average flock size but would introduce huge cyclic extremes in population, which would exacerbate his economic difficulties.

3. Clearly, many changes could be made to the model which would improve its applicability to this situation. Several suggestions are listed below.

a) The model assumes that predators have only one food source (the prey species), which is not realistic in this situation. A term could be added to represent the lumped effects of alternative food sources.

b) The model assumes that prey die only due to predation. A term could be added to represent the lumped effects of other means of death, *e.g.* disease, old age, and severe weather.

c) The model bases reproduction rate upon the number of individuals present. This is reasonable for species subject to asexual reproduction, but for mammals growth would more logically be proportional to the number of pairs , $n_1/2$. Even better, reproduction could be modeled as being proportional to the number of interactions between members of the opposite sex, $(n_1/2)^2$. The model would then become

TABLE 1

Population Dynamics With Trapping

TIME (years)	SHEEP	COYOTES
0.0	2350.0	33.0
0.7	3578.1	33.7
1.4	5193.9	37.0
2.1	6641.0	44.0
2.7	6780.3	54.8
3.4	5226.8	65.6
4.1	3246.9	71.4
4.8	1880.3	71.4
5.5	1144.7	67.9
6.2	779.2	62.8
6.8	606.1	57.3
7.5	539.7	52.0
8.2	545.6	47.2
8.9	618.9	42.8
9.6	776.4	39.1
10.3	1059.3	36.1
11.0	1541.8	34.0
11.6	2335.6	33.0
12.3	3557.1	33.7
13.0	5169.5	36.9
13.7	6626.6	43.9
14.4	6791.7	54.6
15.1	5256.2	65.5
15.8	3272.1	71.4
16.4	1894.9	71.5
17.1	1152.1	67.9
17.8	782.8	62.9
18.5	607.8	57.4
19.2	540.1	52.1
19.9	545.1	47.2
20.5	617.3	42.9
21.2	773.3	39.2
21.9	1054.0	36.2
22.6	1532.9	34.0
23.3	2321.3	33.0
24.0	3536.2	33.7

Population Dynamics Without Trapping

TIME (years)	SHEEP	COYOTES
0.0	2350.0	53.0
0.7	2195.0	52.4
1.4	2091.0	51.5
2.1	2044.3	50.3
2.7	2056.1	49.2
3.4	2124.7	48.2
4.1	2245.2	47.4
4.8	2408.3	47.0
5.5	2596.8	47.0
6.2	2784.2	47.5
6.8	2935.6	48.4
7.5	3015.8	49.5
8.2	3002.2	50.8
8.9	2896.3	52.0
9.6	2724.2	52.8
10.3	2525.2	53.2
11.0	2336.6	53.0
11.6	2185.1	52.3
12.3	2085.3	51.4
13.0	2043.1	50.3
13.7	2059.4	49.1
14.4	2132.2	48.1
15.1	2256.4	47.4
15.8	2422.1	47.0
16.4	2611.6	47.0
17.1	2797.6	47.5
17.8	2944.7	48.4
18.5	3018.2	49.6
19.2	2997.2	50.9
19.9	2885.0	52.1
20.5	2709.4	52.9
21.2	2509.9	53.2
21.9	2323.4	52.9
22.6	2175.5	52.3
23.3	2080.0	51.3
24.0	2042.3	50.2

$$\frac{dn_1}{dt} = a\left(\frac{n_1}{2}\right)^2 - kn_1n_2$$

d) The model assumes an environment shielded from external intervention, *e.g.* urban growth displacing coyotes from an adjacent region into your region. Such factors would be difficult to incorporate into the model explicitly, but their existence adds to the uncertainty of the results.

e) The model predicts unbounded prey growth in the absence of the predator. This is clearly unrealistic as other limits to growth exist, *e.g.* food supply and land area. The specific growth rate term could be modified to approach zero when the population reaches the maximum number which the environment can maintain. For example, if the food supply (F) were taken to be the limiting factor in the absence of predators, the model might become:

$$\frac{dn_1}{dt} = \left(a - \frac{b}{c + F}\right) n_1 \qquad \text{where } \left(\frac{b}{c}\right) = a$$

An additional equation for how F changes with n_1 and n_2 would also be required.

f) The parameter estimates are clearly a large source of uncertainty. A sensitivity analysis could be done on the parameters, and improved estimates could be sought for those having the greatest impact.

For example, the parameters of most interest are those determining the steady state sheep population, *i.e.*, b, q and k. A 20% increase in the estimate of b results in a comparable increase in the steady state population, but the average coyote population is unchanged. Similarly, decreases in q elevate the prey population without affecting the predator population. If the estimates for k and q were increased and decreased by 20% respectively, the prey population would increase by 25.0%, while the predator population would decrease by 16.6%. However, these rather modest changes in parameter and steady state values are accompanied by drastically different behavior in the phase plane representation of population dynamics. As seen in Figure 2, a totally different vision of the effects of trapping emerges when these parameter changes are made. Indeed, even the qualitative trends are inverted, with the larger oscillations in population occurring *before* the trapping operation.

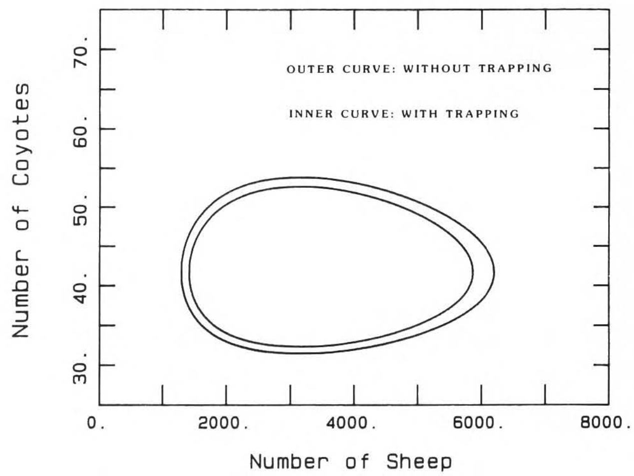

FIGURE 2. Predator/Prey Population Cycles

CONCLUSION

Counterparts to the undamped oscillation examined here in a biological context are readily found in chemical engineering applications. In both cases competing effects may be identified as the underlying cause of the oscillation. In the biological example, the rate of increase in prey population is enhanced by enlarged population size and decreased by encounters with the predator species. Conversely, the growth rate of the predator population is negatively affected by increases in its magnitude and benefited by increased encounters with the prey.

Conceptually similar phenomena may be identified in a temperature-controlled CSTR in which an exothermic decomposition reaction occurs. The rate of heating is elevated by increases in reactant concentration and reactor temperature. Heat exchange coils in the reactor constantly remove heat at a rate proportional to the difference between the reactor and cooling-water temperatures. In a simple control scheme, additional cooling capacity would be engaged whenever the reactor temperature exceeds the setpoint temperature. The rate of the added cooling would be proportional to the deviation from the desired temperature.

Hence, for such a reactor, high temperatures invoke a high cooling rate with concomitant decreases in the reaction (heating) rate, and the temperature falls. In contrast, low temperatures result in low cooling rates and low reaction rate constants. Resulting increases in reactant concentration raise the reaction (heating) rate, and the temperature rises. For certain combinations of system parameters, these competing effects generate limit cycles very similar to those displayed by the predator-prey example. However, a noteworthy distinction may be made between the two types of oscillations: The position of the predator-prey population cycle in the phase plane depends upon the initial population sizes, but for the chemical reactor, the location of the cycle is independent of initial reactor temperature.

ACKNOWLEDGMENT

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