SCALING INITIAL AND BOUNDARY VALUE PROBLEMS

A Tool in Engineering Teaching and Practice

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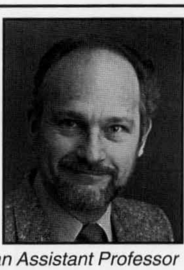
Scaling in the context of this paper refers to the systematic method whereby one nondimensionalizes a system of equations describing a transport and/or chemical reaction process in order to determine the minimum parametric representation; that is, the description of the process in terms of the minimum number of dimensionless groups. This permits assessing how the system of equations can be simplified for very large or very small values of the dimensionless groups. For example, the equations of motion can be appropriately nondimensionalized so that the inertial terms can be neglected for very small Reynolds numbers which is the familiar creeping flow approximation.

Textbooks on transport and chemical reaction processes generally justify simplifying assumptions leading to the creeping flow, boundary layer, penetration theory, plug-flow reactor, etc., equations via *ad hoc* arguments rather than by a systematic approach such as scaling analysis provides. Hence, the student might not see the interrelationship between the various approximations made in describing transport and reactor-design processes such as the analogy between boundary-layer theory in fluid mechanics and penetration theory in heat or mass transfer. Moreover, the *ad hoc* approach to simplifying the equations describing transport and chemical reaction processes does not provide the student with any basis for simplifying more complex problems which are not described in textbooks.



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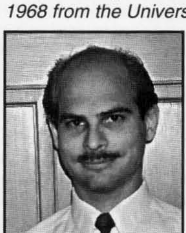

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In an earlier article in this journal, Krantz[1] described how scaling analysis can be used to simplify the initial and boundary value problems encountered in teaching transport phenomena. The present article builds upon this earlier paper by showing how scaling analysis can be used to justify the quasi-steady-state approximation and by demonstrating the application of this technique to simplifying problems involving entry-region flows, moving boundaries, porous media flows, and mass transfer with chemical reaction.

THE SCALING ANALYSIS TECHNIQUE

Scaling analysis can be reduced to the following stepwise procedure:

- 1. Write down the dimensional differential equations and their initial and boundary conditions appropriate to the transport or reactor-design process being considered.
- 2. Form dimensionless variables by introducing unspecified scale factors for each dependent and independent variable; this also may involve introducing unspecified reference factors for some variables whose values we seek to normalize to zero.
- 3. Introduce these dimensionless variables into the describing differential equations and their initial and boundary conditions.
- 4. Divide through by the dimensional coefficient of one of the terms (preferably one which will be retained) in each of the describing equations and their initial and boundary conditions.
- 5. Determine the scale and reference factors by insuring that the principal terms in the describing equations are of order one; identifying the principal terms is dependent on the particular conditions for which the scaling is being done (*e.g.*, a highly viscous flow, a conductive heat-transfer process, etc.; this step may require introducing a "region-of-influence" wherein the dependent variable(s) goes through a characteristic change in value).
- 6. The preceding steps result in the minimum parametric representation of the problem (*i.e.*, in terms of the minimum number of dimensionless groups); appropriate simplification of the describing equations can now be explored for very small or very large values of these dimensionless groups.

Application of scaling analysis now will be illustrated via several example problems. The first problem will be shown in detail to illustrate the scaling method, whereas the other examples will only be outlined.

EXAMPLE PROBLEMS

1. Laminar Flow Between Parallel Plates

Figure 1 shows a schematic of steady-state, fully developed, laminar flow between two infinitely wide parallel plates. The lower plate is stationary and the upper plate moves at a constant velocity V_p. This flow is also subject to a constant axial pressure gradient such that $\Delta P > 0$. We seek to determine the conditions for which the effect of the upper plate velocity V_p can be neglected.

The appropriate equations of motion and their boundary conditions are given by

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{d^2 v_x}{dy^2} \tag{1.1}$$

$$0 = -\frac{\partial P}{\partial y} + \rho g \tag{1.2}$$

$$v_x = 0 \quad \text{at} \quad y = 0 \tag{1.3}$$

$$v_x = V_p \quad \text{at} \quad y = H \tag{1.4}$$

Equation (1.2) can be integrated and combined with Eq. (1.1) to obtain

$$0 = \frac{\Delta P}{L} + \mu \frac{d^2 v_x}{dy^2} \tag{1.5}$$

where $\Delta P \equiv P|_{x=0} - P|_{x=L}$. Define the following dimensionless variables

$$v_x^* \equiv \frac{v_x}{U_s} \quad \text{and} \quad y^* \equiv \frac{y}{y_s} \tag{1.6}$$

Substituting these into Eqs. (1.3), (1.4) and (1.5) then yields

$$0 = \frac{\Delta P}{L} + \frac{\mu U_s}{y_s^2} \frac{d^2 v_x^*}{dy^{*2}} \tag{1.7}$$

$$U_s v_x^* = 0 \quad \text{at} \quad y_s y^* = 0 \tag{1.8}$$

$$U_s v_x^* = V_p \quad \text{at} \quad y_s y^* = H \tag{1.9}$$

Since the viscous term in Eq. (1.7) must be retained in order to satisfy the two no-slip conditions at the solid boundaries, divide through by its dimensional coefficient. Similarly, in the two boundary conditions divide through by the dimensional coefficient of the dimensionless dependent variable. This yields

$$0 = \frac{y_s^2 \Delta P}{\mu U_s L} + \frac{d^2 v_x^*}{dy^{*2}} \tag{1.10}$$

$$v_x^* = 0 \quad \text{at} \quad y^* = 0 \tag{1.11}$$

$$v_x^* = \frac{V_p}{U_s} \quad \text{at} \quad y^* = \frac{H}{y_s} \tag{1.12}$$

Since we are scaling this problem for conditions such that the flow is caused principally by the pressure gradient, we balance the pressure force with the viscous term in Eq. (1.10) as follows

$$\frac{y_s^2 \Delta P}{\mu U_s L} = 1 \tag{1.13}$$

Note that this insures that the magnitude of the dimensionless derivative, $d^2 v_x^*/dy^{*2}$, is of order one. Furthermore, the dimensionless independent variable y^* will be bounded of order one if we demand that

$$\frac{H}{y_s} = 1 \quad \Rightarrow \quad y_s = H \tag{1.14}$$

Hence, from Eq. (1.13) we obtain

$$U_s = \frac{H^2 \Delta P}{\mu L} \tag{1.15}$$

Note that this velocity scale is directly proportional to the maximum velocity for flow between two flat plates driven only by a pressure gradient. This scaling insures that the dimensionless velocity goes through a change of order one over a dimensionless distance of order one. Note that "a change of order one" implies that the dimensionless variable goes from its minimum value of zero to its maximum value which has a magnitude of order one.

Our dimensionless equations now become

$$0 = 1 + \frac{d^2 v_x^*}{dy^{*2}} \tag{1.16}$$

$$v_x^* = 0 \quad \text{at} \quad y^* = 0 \tag{1.17}$$

$$v_x^* = \frac{V_p \mu L}{H^2 \Delta P} \quad \text{at} \quad y^* = 1 \tag{1.18}$$

Hence, in order to ignore the effect of the moving upper plate on the flow relative to that of the imposed pressure gradient, we must satisfy the criterion that

$$\frac{V_p \mu L}{H^2 \Delta P} \ll 1 \tag{1.19}$$

One could also scale this problem for conditions such that the flow is caused principally by the upper moving boundary. In this case, we determine our velocity scale from Eq. (1.12) and obtain

$$U_s = V_p \tag{1.20}$$

Hence, our dimensionless equations become

$$0 = \frac{H^2 \Delta P}{\mu V_p L} + \frac{d^2 v_x^*}{dy^{*2}} \tag{1.21}$$

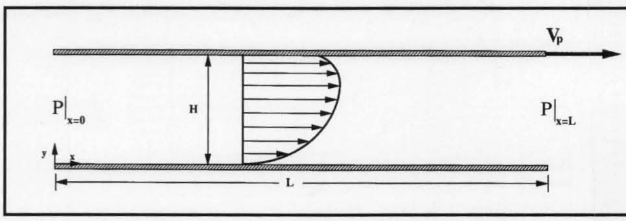

***Figure 1.** Schematic of steady-state, full developed, laminar flow between two infinitely wide parallel plates; the lower plate is stationary and the upper plate moves at a constant velocity V_p.*

In order to ignore the effect of the pressure gradient on the flow, we must satisfy the criterion

$$\frac{H^2 \Delta P}{\mu V_p L} \ll 1 \tag{1.22}$$

This simple example problem can be solved analytically, which permits assessing the error incurred by ignoring the plate velocity under the condition that Eq. (1.19) is satisfied, or ignoring the pressure gradient under the condition that Eq. (1.22) is satisfied. For example, if

$$\frac{V_p \mu L}{H^2 \Delta P} \le 0.1$$

comparison with the exact analytical solution shows that we will incur a maximum error of 20% in the drag at the wall. A maximum error of 2% in the drag at the wall is implied by

$$\frac{V_p \mu L}{H^2 \Delta P} \le 0.01$$

One sees that scaling not only provides the criteria for simplifying the equations describing transport and chemical reaction processes, but also provides a measure of the error incurred in making these simplifications. This illustrates the advantages of scaling the principal dimensionless terms to be of order one; that is, the error incurred is of the same order as the dimensionless group which must be small to ignore the term in question.

2. Entry Region for Flow Between Parallel Plates

Figure 2 shows a schematic of pressure-driven, steady-state, laminar entry-region flow between two infinitely wide stationary parallel plates; the flow velocity at the entrance is assumed to be constant at a value $v_x = \overline{V}$. We seek to determine the condition required to attain fully developed laminar flow. This example will illustrate how to handle a boundary condition that introduces an unknown region-of-influence [in this case, $\delta(x)$]. Scaling will allow us to determine the functional form of $\delta(x)$ to within a multiplicative constant of order one.

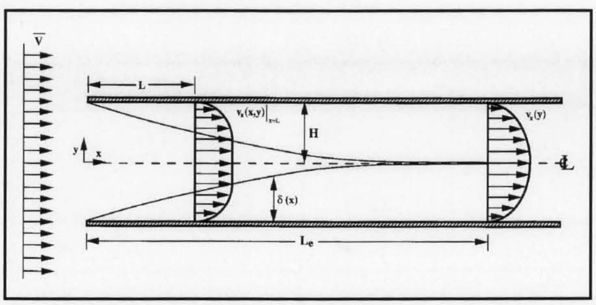

***Figure 2.** Schematic of pressure-driven, steady-state, laminar entry-region flow between two infinitely wide stationary parallel plates.*

The appropriate equations-of-motion and their boundary conditions are given by

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial x^2} + \mu \frac{\partial^2 v_x}{\partial y^2} \tag{2.1}$$

$$\rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} = -\frac{\partial P}{\partial y} + \mu \frac{\partial^2 v_y}{\partial x^2} + \mu \frac{\partial^2 v_y}{\partial y^2} \tag{2.2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{2.3}$$

$$v_x = \overline{V} \quad \text{and} \quad v_{y=0} \quad \text{at} \quad x = 0 \tag{2.4}$$

$$v_x = v_x(y) \quad \text{and} \quad v_y = v_y(y) \quad \text{at} \quad x = L \tag{2.5}$$

$$v_x = 0 \quad \text{and} \quad v_y = 0 \quad \text{at} \quad y = \pm H \tag{2.6}$$

$$v_x = v_x(x) \quad \text{and} \quad v_y = v_y(x) \quad \text{at} \quad y = \pm(H - \delta) \tag{2.7}$$

We have elected to use the complete form of the two-dimensional equations-of-motion rather than the boundary-layer approximation. The manner in which the latter can be derived via scaling analysis is discussed by Krantz.[1] The boundary condition given by Eq. (2.7) introduces the region of influence variable $\delta(x)$ which defines the boundary layer thickness near the wall wherein the viscous effects are confined and hence in which the development of the velocity profile occurs. Equation (2.5) is included for completeness and indicates that the velocity profiles must be specified at some downstream point x = L in order to solve the complete form of these equations. Equation (2.7) merely indicates that there is acceleration of the core fluid outside of the boundary layer. It is not necessary to specify any boundary conditions on the pressure since specifying the constant inlet velocity determines the required pressure gradient.

Define the following dimensionless variables:

$$v_x^* \equiv \frac{v_x}{U_s}; \quad v_y^* \equiv \frac{v_y}{V_s}; \quad P^* \equiv \frac{P}{P_s}; \quad x^* \equiv \frac{x}{x_s}; \quad y^* \equiv \frac{y - y_r}{y_s} \tag{2.8}$$

We have introduced a reference scale y_r in the definition of y^* in order to reference this dimensionless variable to zero at the wall; the symmetry of this problem permits considering only the region $-H \le y \le 0$. Substituting these dimensionless variables into Eqs. (2.1) through (2.7) and dividing through by the dimensional coefficient of one of the principal terms in each equation then yields

$$\frac{\rho U_s y_s^2}{\mu x_s} v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{\rho V_s y_s}{\mu} v_y^* \frac{\partial v_x^*}{\partial y^*} = -\frac{P_s y_s^2}{\mu U_s x_s} \frac{\partial P^*}{\partial x^*} + \frac{y_s^2}{x_s^2} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}} \tag{2.9}$$

$$\frac{\rho U_s y_s^2}{\mu x_s} v_x^* \frac{\partial v_y^*}{\partial x^*} + \frac{\rho V_s y_s}{\mu} v_y^* \frac{\partial v_y^*}{\partial y^*} = -\frac{P_s y_s}{\mu V_s} \frac{\partial P^*}{\partial y^*} + \frac{y_s^2}{x_s^2} \frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{\partial^2 v_y^*}{\partial y^{*2}} \tag{2.10}$$

$$\frac{\partial v_x^*}{\partial x^*} + \frac{V_s x_s}{U_s y_s} \frac{\partial v_y^*}{\partial y^*} = 0 \tag{2.11}$$

$$v_x^* = \frac{\overline{V}}{U_s} \quad \text{and} \quad v_y^* = 0 \quad \text{at} \quad x^* = 0 \tag{2.12}$$

$$v_x^* = v_x^*(y^*) \quad \text{and} \quad v_y^* = v_y^*(y^*) \quad \text{at} \quad x^* = \frac{L}{x_s} \tag{2.13}$$

$$v_x^* = 0 \quad \text{and} \quad v_y^* = 0 \quad \text{at} \quad y^* = \frac{-H - y_r}{y_s} \tag{2.14}$$

$$v_x^* = v_x^*(x^*) \quad \text{and} \quad v_y^* = v_y^*(x^*) \quad \text{at} \quad y^* = \frac{-(H-\delta)-y_r}{y_s} \tag{2.15}$$

We can normalize y^* between the values of 0 and 1 by requiring the following:

$$\frac{-H-y_r}{y_s} = 0 \quad \Rightarrow \quad y_r = -H \tag{2.16}$$

and

$$\frac{-(H-\delta)-y_r}{y_s} = 1 \quad \Rightarrow \quad y_s = \delta \tag{2.17}$$

In order for the dimensionless axial velocity and axial coordinate to be bounded between 0 and 1, we require that

$$\frac{\overline{V}}{U_s} = 1 \quad \Rightarrow \quad U_s = \overline{V} \tag{2.18}$$

$$\frac{L}{x_s} = 1 \quad \Rightarrow \quad x_s = L \tag{2.19}$$

Since this is a developing flow, both terms in the dimensionless continuity equation should be of order one; hence, we require that

$$\frac{V_s x_s}{U_s y_s} = \frac{U_s L}{\overline{V}\delta} = 1 \quad \Rightarrow \quad V_s = \overline{V}\frac{\delta}{L} \tag{2.20}$$

Since this is a pressure-driven, viscous flow, the dimensionless pressure term should be of the same order as the principal viscous term, $\partial^2 v_x^* / \partial y^{*2}$; hence, we require that

$$\frac{P_s y_s^2}{\mu U_s x_s} = \frac{P_s \delta^2}{\mu \overline{V} L} = 1 \quad \Rightarrow \quad P_s = \frac{\mu \overline{V} L}{\delta^2} \tag{2.21}$$

Substituting these values of the scale and reference factors yields the minimum parametric representation given by

$$Re\frac{\delta}{L}v_x^*\frac{\partial v_x^*}{\partial x^*} + Re\frac{\delta}{L}v_y^*\frac{\partial v_x^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{\delta^2}{L^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}} \tag{2.22}$$

$$Re\frac{\delta}{L}v_x^*\frac{\partial v_y^*}{\partial x^*} + Re\frac{\delta}{L}v_y^*\frac{\partial v_y^*}{\partial y^*} = -\frac{L^2}{\delta^2}\frac{\partial P^*}{\partial y^*} + \frac{\delta^2}{L^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{\partial^2 v_y^*}{\partial y^{*2}} \tag{2.23}$$

$$v_x^* = 1 \quad \text{and} \quad v_y^* = 0 \quad \text{at} \quad x^* = 0 \tag{2.24}$$

$$v_x^* = v_x^*(y^*) \quad \text{and} \quad v_y^* = v_y^*(y^*) \quad \text{at} \quad x^* = 1 \tag{2.25}$$

$$v_x^* = 0 \quad \text{and} \quad v_y^* = 0 \quad \text{at} \quad y^* = 0 \tag{2.26}$$

$$v_x^* = v_x^*(x^*) \quad \text{and} \quad v_y^* = v_y^*(x^*) \quad \text{at} \quad y^* = 1 \tag{2.27}$$

where $Re \equiv \delta\rho\overline{V}/\mu$.

Since this is a developing flow, the intertial terms must be of the same magnitude as the pressure and principal viscous term, $\partial^2 v_x^* / \partial y^{*2}$, in Eq. (2.22); hence, we require that

$$Re\frac{\delta}{L} = 1 \quad \Rightarrow \quad \delta = \left(\frac{L\mu}{\rho\overline{V}}\right)^{1/2} \tag{2.28}$$

Hence, we see that scaling analysis gives us the boundary-layer thickness to within a multiplicative constant of order one. Scaling also can provide a reliable estimate of the entry length L_e required for the flow to become fully developed; this is obtained by setting $\delta = H$ in Eq. (2.28) to obtain

$$L_e \approx \frac{\rho\overline{V}H^2}{\mu} \tag{2.29}$$

A boundary-layer analysis yields the following solution for the entry length:[2]

$$L_e = 0.16\frac{\rho\overline{V}H^2}{\mu} \tag{2.30}$$

Hence, we see that scaling analysis gives the correct result for the entry length to within a multiplicative constant of order one.

3. Flow Through a Porous Medium in a Cylindrical Tube

Figure 3 shows a schematic of pressure-driven, steady-state flow of a fluid having viscosity μ through a porous medium having a permeability K confined in a horizontal tube of radius R and length L. We seek to determine the criterion for ignoring the drag on the tube wall when determining the volumetric flow rate and the thickness of the region of influence near the wall wherein this approximation is not valid. The appropriate forms of the equations of motion and boundary conditions are given by[3]

$$0 = \frac{\Delta P}{L} - \frac{\mu}{K}v_z + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) \tag{3.1}$$

$$v_z = 0 \quad \text{at} \quad r = R \tag{3.2}$$

$$\frac{dv_z}{dr} = 0 \quad \text{at} \quad r = 0 \tag{3.3}$$

where ΔP is the pressure drop across the length of the tube. The second term on the right in Eq. (3.1) is referred to as the Darcy flow term.

Define the following dimensionless variables:

$$v_z^* \equiv \frac{v_z}{W_s} \quad \text{and} \quad r^* \equiv \frac{r}{r_s} \tag{3.4}$$

Introducing these dimensionless variables into Eqs. (3.1) through (3.3) and dividing through by the dimensional coefficient of one of the principal terms in each equation yields

$$0 = 1 - \frac{\mu W_s L}{K\Delta P}v_z^* + \frac{\mu W_s L}{r_s^2\Delta P}\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\frac{dv_z^*}{dr^*}\right) \tag{3.5}$$

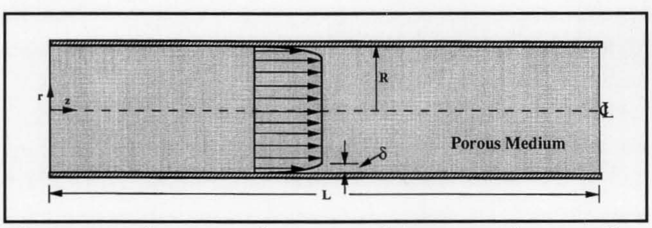

***Figure 3.** Schematic of pressure-driven, steady-state flow of a viscous fluid through a porous medium confined within a horizontal tube of radius R and length L.*

$$v_z^* = 0 \quad \text{at} \quad r^* = \frac{R}{r_s} \tag{3.6}$$

$$\frac{dv_z^*}{dr^*} = 0 \quad \text{at} \quad r^* = 0 \tag{3.7}$$

If the porous medium is the principal resistance to flow, then we require that

$$\frac{\mu W_s L}{K\Delta P} = 1 \quad \Rightarrow \quad W_s = \frac{K\Delta P}{\mu L} \tag{3.8}$$

If the drag at the tube wall were important, the velocity would change significantly over a length scale of the same order as the tube radius R. Hence, in order to assess the effect of the drag at the tube wall on the volumetric flow rate, we determine our length scale by demanding that

$$\frac{R}{r_s} = 1 \quad \Rightarrow \quad r_s = R \tag{3.9}$$

which bounds r^* between 0 and 1. Substituting Eqs. (3.8) and (3.9) into Eq. (3.5) then yields

$$0 = 1 - v_z^* + \frac{K}{R^2}\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\frac{dv_z^*}{dr^*}\right) \tag{3.10}$$

Hence, in order to ignore the drag at the tube wall, we require that

$$\frac{K}{R^2} \ll 1 \tag{3.11}$$

Ignoring the viscous term when the criterion given by Eq. (3.11) is satisfied yields a very accurate prediction for the volumetric flow rate through the porous media; but it will not predict the velocity profile accurately throughout the flow since clearly the velocity must be zero at the tube wall. This implies that there is a region of influence having a thickness δ near the tube wall wherein the last term in Eq. (3.5) is of the same magnitude as the pressure and Darcy flow terms. Within this boundary layer the radial coordinate must be scaled with δ rather than R to insure that the dimensionless velocity gradient is of order one. The thickness of this boundary layer region can be determined by exploring the conditions for which the last term in Eq. (3.5) is also of order one; that is, when

$$\frac{\mu W_s L}{\delta^2 \Delta P} = \frac{K}{\delta^2} = 1 \quad \Rightarrow \quad \delta = \sqrt{K} \tag{3.12}$$

Hence the thickness of the region of influence wherein the walls of the tube influence the flow through the porous medium is of the order $\sqrt{K}$.

4. Unsteady-State Heat Conduction with Phase Change

Figure 4 shows a schematic of unsteady-state, one-dimensional heat conduction into an initially frozen semi-infinite slab of soil subjected to a constant temperature T_0 ($T_0 > T_f$) at its surface. The soil is assumed to be initially at its freezing temperature T_f. We seek to determine when this heat-transfer process can be approximated by quasi-steady-state heat conduction. This example will illustrate how to apply scaling analysis to an unsteady-state moving boundary problem. The appropriate forms of the energy equation, initial, boundary, and auxiliary conditions are given by

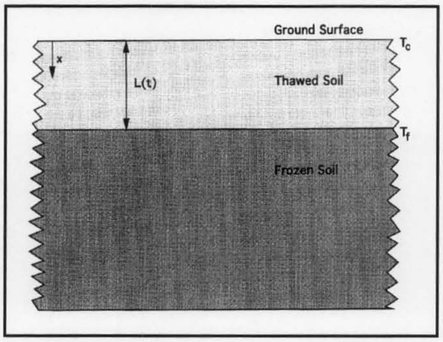

Figure 4. *Schematic of unsteady-state, one-dimensional heat conduction into an initially frozen semi-infinite slab of soil subjected to a constant temperature T_0 at its surface.*

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{4.1}$$

$$T = T_f \quad \text{and} \quad L = 0 \quad \text{at} \quad t = 0 \tag{4.2}$$

$$T = T_0 \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0 \tag{4.3}$$

$$T = T_f \quad \text{at} \quad x = L(t) \tag{4.4}$$

$$\frac{dL}{dt} = -\frac{k}{\rho\lambda}\frac{\partial T}{\partial x} \quad \text{at} \quad x = L(t) \tag{4.5}$$

where α, k, and ρ are the thermal diffusivity, thermal conductivity, and density of the frozen soil, respectively, and λ is the latent heat of fusion of water; L(t) is the instantaneous thaw depth.

Define the following dimensionless variables:

$$T^* \equiv \frac{T - T_r}{T_s}, \quad x^* \equiv \frac{x}{x_s} \quad \text{and} \quad t^* \equiv \frac{t}{t_s} \tag{4.6}$$

Introducing these dimensionless variables into Eqs. (4.1) through (4.5) and dividing through by the dimensional coefficient of one of the principal terms in each equation then yields

$$\frac{x_s^2}{\alpha t_s}\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}} \tag{4.7}$$

$$T^* = \frac{T_f - T_r}{T_s} \quad \text{at} \quad t^* = 0 \tag{4.8}$$

$$T^* = \frac{T_0 - T_r}{T_s} \quad \text{at} \quad x^* = 0 \tag{4.9}$$

$$T^* = \frac{T_f - T_r}{T_s} \quad \text{at} \quad x^* = L^* \tag{4.10}$$

$$\frac{dL^*}{dt^*} = -\frac{kT_s t_s}{\rho\lambda x_s^2}\frac{\partial T^*}{\partial x^*} \quad \text{at} \quad x^* = L^* \tag{4.11}$$

In order to insure that the dimensionless temperature is bounded between 0 and 1, we demand that

$$T^* = \frac{T_f - T_r}{T_s} = 0 \quad \Rightarrow \quad T_r = T_f \tag{4.12}$$

$$T^* = \frac{T_0 - T_r}{T_s} = 1 \quad \Rightarrow \quad T_s = T_0 - T_f \tag{4.13}$$

In unsteady problems for which we seek to determine the applicability of the quasi-steady-state approximation, the time scale is the observation time, t_o; that is

$$t_s = t_o \tag{4.14}$$

Since the two terms in Eq. (4.11) must balance for a moving boundary problem, we require that

$$\frac{kT_s t_s}{\rho\lambda x_s^2} = \frac{k(T_0 - T_f)t_o}{\rho\lambda x_s^2} = 1 \quad \Rightarrow \quad x_s = \left[\frac{k(T_0 - T_f)t_o}{\rho\lambda}\right]^{1/2} \tag{4.15}$$

Note this length scale insures that the characteristic rate of heat removal by conduction balances the heat released owing to melting a thickness x_s of ice. Equation (4.7) then assumes the form

$$\frac{k(T_0 - T_f)}{\rho\lambda\alpha}\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}} \tag{4.16}$$

Hence, the criterion for invoking the quasi-steady-state approximation is given by

$$\frac{k(T_0 - T_f)}{\rho\lambda\alpha} << 1 \tag{4.17}$$

5. Laminar Flow with Heterogeneous Reaction at the Wall

Figure 5 shows a schematic of steady-state laminar flow in a tube of radius R at which a solute A contained in the fluid undergoes a first-order irreversible reaction along length L. We seek to determine the conditions required to justify two different approximations: to assume that the reaction causes total depletion of A at the pipe wall; and, to make the classical "plug flow reactor" approximation for which the radial concentration gradient is ignored and the flow is assumed to be pluglike and equal to the average velocity.

The appropriate form of the conservation of species equation and its boundary conditions is given by

$$v_z \frac{\partial c_A}{\partial z} = D_{AB}\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial c_A}{\partial r}\right) \tag{5.1}$$

$$c_A = c_{A0} \quad \text{at} \quad z = 0 \tag{5.2}$$

$$\frac{\partial c_A}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{5.3}$$

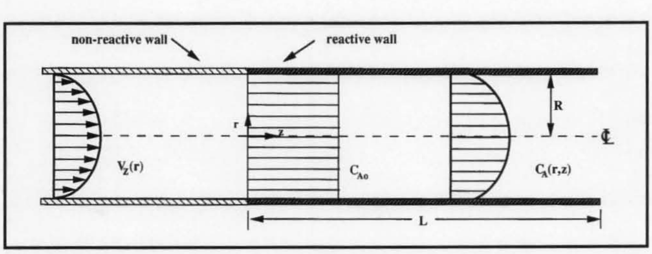

Figure 5. *Schematic of steady-state laminar flow in a tube of radius R at which a solute A contained in the fluid undergoes a first-order irreversible reaction along length L.*

$$-D_{AB}\frac{\partial c_A}{\partial r} = k_1 c_A \quad \text{at} \quad r = R \quad 0 \le z \le L \tag{5.4}$$

in which D_{AB} is the binary diffusion coefficient, c_{A0} is the initial concentration of the reactant A, k_1 is the first-order reaction-rate constant, and v_z is the laminar flow velocity given by

$$v_z = \frac{3}{2}\overline{V}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{5.5}$$

where $\overline{V}$ is the average velocity. Note that we have ignored axial diffusion relative to convection of species A.

Introduce the following dimensionless variables:

$$c_A^* \equiv \frac{c_A}{c_s}, \qquad r^* \equiv \frac{r}{r_s} \qquad \text{and} \qquad z^* \equiv \frac{z}{z_s} \tag{5.6}$$

and divide through by the dimensional coefficient of one term to obtain

$$\frac{3}{2}\left[1 - \left(\frac{r_s}{R}\right)^2 r^{*2}\right]\frac{\partial c_A^*}{\partial z^*} = \frac{D_{AB}}{\overline{V}}\frac{z_s}{r_s^2}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial c_A^*}{\partial r^*}\right) \tag{5.7}$$

$$c_A^* = \frac{c_{A0}}{c_s} \quad \text{at} \quad z^* = 0 \tag{5.8}$$

$$\frac{\partial c_A^*}{\partial r^*} = 0 \quad \text{at} \quad r^* = 0 \tag{5.9}$$

$$-\frac{\partial c_A^*}{\partial r^*} = \frac{k_1 r_s}{D_{AB}} c_A^* \quad \text{at} \quad r^* = \frac{R}{r_s} \quad 0 \le z^* \le \frac{L}{z_s} \tag{5.10}$$

The dimensionless groups suggest the following choices for the scale factors:

$$\frac{c_{A0}}{c_s} = 1 \quad \Rightarrow \quad c_s = c_{A0} \tag{5.11}$$

$$\frac{r_s}{R} = 1 \quad \Rightarrow \quad r_s = R \tag{5.12}$$

$$\frac{L}{z_s} = 1 \quad \Rightarrow \quad z_s = L \tag{5.13}$$

Hence, our describing equations become

$$\frac{3}{2}\left[1 - r^{*2}\right]\frac{\partial c_A^*}{\partial z^*} = \frac{D_{AB}}{\overline{V}}\frac{L}{R^2}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial c_A^*}{\partial r^*}\right) \tag{5.14}$$

$$c_A^* = 1 \quad \text{at} \quad z^* = 0 \tag{5.15}$$

$$\frac{\partial c_A^*}{\partial r^*} = 0 \quad \text{at} \quad r^* = 0 \tag{5.16}$$

$$-\frac{\partial c_A^*}{\partial r^*} = \frac{k_1 R}{D_{AB}} c_A^* \quad \text{at} \quad r^* = 1 \quad 0 \le z^* \le 1 \tag{5.17}$$

If $k_1R/D_{AB} >> 1$, then we must have $c_A^* \approx 0$ in order to assure that $\partial c_A^* / \partial r^*$ is of order one at $r^* = 1$. Hence, for this limiting case corresponding to a very fast heterogeneous reaction, the boundary condition given by Eq. (5.17) can be replaced by

$$c_A^* \approx 0 \quad \text{at} \quad r^* = 1 \tag{5.18}$$

If, in constrast, $k_1R/D_{AB} << 1$, then since c_A^* is of order one,

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CONCLUSION

Adsorption represents an important unit operation in the chemical industry. It is a fertile area with research opportunities in both fundamental and applied aspects. For those students who are interested in pursuing research in this area, the course is designed to provide sufficient fundamental background and an appreciation of the status of current research efforts in different areas. After taking the course, the graduate students are in a better position to identify an area of research interest. For others, it provides an understanding of the fundamentals of adsorption and its place in industrial applications for separation and purification. The course has been well received by the students.

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we must have $\partial c_A^* / \partial r^* << 1$ at $r^* = 1$. But $\partial c_A^* / \partial r^*$ is largest at $r^* = 1$; hence $\partial c_A^* / \partial r^* << 1$ throughout the tube, and we conclude that $c_A^* \approx c_A(z^*)$. Since the radial concentration gradient is negligible, we can incorporate the heterogeneous reaction term directly into the species mass balance to obtain the classical plug flow reaction equation

$$\overline{v}\frac{dc_A}{dz} = -\frac{2k_1}{R}c_A \quad \text{for} \quad 0 \le z \le L \qquad (5.19)$$

$$c_A = c_{A_0} \quad \text{at} \quad z = 0 \qquad (5.20)$$

where $\overline{v}$ is the average velocity.

SUMMARY

Hopefully these five examples have convinced the reader that the systematic approach to scaling analysis described here has real utility in teaching transport-related engineering courses as well as in engineering practice. Additional examples of scaling analysis were given in the earlier article by Krantz.[1] Reprints of the latter article can be obtained by contacting the authors.

NOMENCLATURE

- c_A molar concentration of component A
- c_{A_0} initial molar concentration of component A
- D_{AB} binary diffusion coefficient of A in B
- g gravitational acceleration
- H spacing or half-spacing between parallel plates
- k thermal conductivity
- k_1 first-order heterogeneous reaction-rate constant
- K Darcy permeability of porous media
- L length of parallel plates or cylindrical tube
- L_e entry length to achieve fully developed laminar flow
- P pressure
- ΔP pressure drop over length L
- r radial coordinate in cylindrical coordinate system
- R radius of cylindrical tube
- Re Reynolds number
- t time
- T temperature
- T_f freezing temperature
- T_0 surface temperature
- U_s scale for velocity component in x-direction
- v_i velocity component in the i-direction
- $\overline{v}$ mass-average velocity
- V_p velocity of the plate
- V_s scale for velocity component in y-direction
- W_s scale for velocity component in z-direction
- x,y rectangular coordinates
- z axial coordinate in cylindrical coordinate system

Subscripts

- r denotes a reference factor
- s denotes a scale factor

Superscripts

- * denotes a dimensionless variable

Greek

- α thermal diffusivity
- δ boundary-layer or region of influence thickness
- λ latent heat of fusion of water
- μ shear viscosity
- ρ mass density

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