LINEAR UNSTEADY TRANSPORT PROBLEMS WHEN THERE IS AN INITIAL STEADY STATE

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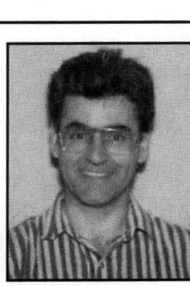
This article will attempt to share a way of setting up linear unsteady-state transport models in beginning graduate and upper-level undergraduate transport phenomena courses, a method that is more general than the one presented in texts and published articles. When setting up a transient problem in momentum, heat, or mass transport, the usual formulation states that before a change in the boundaries or in the driving force takes place, the time- and space-dependent variable is at some homogeneous value in the space domain of interest.[1-4] Thus, in fluid mechanics we say that the velocity is zero throughout when suddenly either a pressure gradient is applied or one of the boundaries is set in motion. Similarly, in heat and mass transfer, the space domain in question is allowed to originally have a certain space-independent temperature (molar concentration) before a new value is imposed on its boundaries, or alternatively the heat (moles) generation experiences some change.

In the following two representative examples, it will be shown how, when the initial condition is not constant with respect to the spatial variable(s) but is instead some steady-state profile, the resulting linear differential equations and initial and boundary conditions can be made to be identical to those of the constant-initial-condition case through the use of a "deviation" dependent variable. The first example is a classic driving-force-change problem, while in the second example the change is imposed on the boundary.



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UNSTEADY HAGEN-POISEUILLE FLOW

For the flow of a liquid of constant density ρ and viscosity μ in a tube of length L and radius R, when the fluid is at rest and suddenly a constant pressure gradient, $(p_0\text{-}p_1)/L$, is applied, the differential equation in dimensionless form is[1]

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) \tag{1}$$

where

$$\phi = v/\left[(p_0 - p_1)R^2/4\mu L\right]$$
$$\xi = r/R$$
$$\tau = t\mu/\rho R^2$$

and v=v(r,t) is the velocity of the liquid in the axial direction of the tube. For this partial differential equation (PDE), the dimensionless initial and boundary conditions are

$$\phi(\tau = 0) = 0; \qquad \phi(\xi = 0) = \text{finite}; \qquad \phi(\xi = 1) = 0$$

Now, instead of having a zero initial condition, let us state that the fluid is originally at a steady state, which we will henceforth refer to as the "original steady-state velocity profile," $\phi_{ss1}(\xi)$. Let the driving force at the original steady state be a pressure gradient $(p_0 - p_1)/L$ and at t=0 the pressure gradient is step-changed to $(p_0 - p_2)/L$. The differential equation is[1]

$$\rho\frac{\partial v}{\partial t} = \frac{p_0 - p_2}{L} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{2}$$

with initial and boundary conditions of

$$\phi(\tau = 0) = \phi_{ss1} = 1 - \xi^2; \qquad \phi(\xi = 0) = \text{finite}; \qquad \phi(\xi = 1) = 0$$

Equation 2 can also be written as

$$\rho\frac{\partial v}{\partial t} = \frac{p_0 - p_1}{L} + \frac{p_1 - p_2}{L} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{3}$$

If we define a "deviation" velocity w, as the v's departure from the original steady-state velocity, $w = v - v_{ss1}$, and then substitute $v=w+v_{ss1}$ in Eq. (3), the following PDE is obtained:

$$\rho\frac{\partial w}{\partial t}=\frac{p_1-p_2}{L}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{p_0-p_1}{L}+\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_{ss1}}{dr}\right) \quad (4)$$

The last two terms in Eq. (4), however, add up to zero because the ordinary differential equation for the original steady state is

$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_{ss1}}{dr}\right)+\frac{p_0-p_1}{L}=0 \quad (5)$$

If we now define the dimensionless deviation velocity, $\theta(\xi,\tau)$, as

$$\theta(\xi,\tau)=\frac{w(r,t)}{(p_1-p_2)R^2/4\mu L} \quad (6)$$

then Eq. (4) takes the form

$$\frac{\partial\theta}{\partial\tau}=4+\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) \quad (7)$$

which is identical to Eq. (1) and also has the same initial and boundary conditions

$$\theta(\tau=0)=0; \quad \theta(\xi=0)=\text{finite}; \quad \theta(\xi=1)=0$$

and has as solution for $\theta(\xi,\tau)$, Eq. 4.1-40 in Bird, Stewart, and Lightfoot.[1]

The heat transfer analogue of Hagen-Poiseuille flow is the problem of heat conduction in an electrically heated wire or coil.[1] For the unsteady case, the problem statement lets the wire or coil operate at some original steady state under a constant heat source S_{e1}. Then at t=0, the heat source is step-changed to S_{e2}. The resulting treatment and differential equations are identical to the ones for the unsteady Poiseuille flow.

UNSTEADY DIFFUSION-REACTION IN A SPHERE

The steady-state differential equation for diffusion and first-order chemical reaction in a sphere, with constant diffusivity $\mathcal{D}$ and reaction rate constant k, is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dC}{d\xi}\right)-\beta^2C=0 \quad (8)$$

where C is the concentration of the reactant, $\xi=r/R$, and $\beta=R\sqrt{k/\mathcal{D}}$. If the surface concentration of the reactant is kept constant at C_0, the two boundary conditions are

$$C(\xi=0)=\text{finite}; \quad C(\xi=1)=C_0$$

If we are interested in the transient response $C(\xi,\tau)$ when the catalyst is originally free of reactant, and suddenly a surface concentration C_0 is imposed, the PDE is

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial C}{\partial\xi}\right)-\beta^2C=\frac{\partial C}{\partial\tau} \quad (9)$$

with initial and boundary conditions

$$C(\tau=0)=0; \quad C(\xi=0)=\text{finite}; \quad C(\xi=1)=C_0$$

where $\tau=t\mathcal{D}/R^2$ is the dimensionless time.

Now, let us suppose that the catalyst is operating at some original steady state and has a concentration profile C_{ss1} under a constant surface concentration C_1, when suddenly the surface concentration is raised to C_2. The PDE will still be given by Eq. (9), but the conditions will be

$$C(\tau=0)=C_{ss1}(\xi)=C_1\sinh\beta\xi/[\xi\sinh\beta]; \quad C(\xi=0)=\text{finite}; \quad C(\xi=1)=C_2$$

Defining a deviation concentration $\Gamma(\xi,\tau)=C(\xi,\tau)-C_{ss1}(\xi)$, we may substitute C in Eq. (9) by $C_{ss1}+\Gamma$. The equation thus becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dC_{ss1}}{d\xi}\right)-\beta^2C_{ss1}+\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Gamma}{\partial\xi}\right)-\beta^2\Gamma=\frac{\partial\Gamma}{\partial\tau} \quad (10)$$

with the first two terms adding up to zero because of Eq. (8). If we define $\Gamma_0=C_2-C_1$, the PDE for $\Gamma(\xi,\tau)$ and its conditions are

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Gamma}{\partial\xi}\right)-\beta^2\Gamma=\frac{\partial\Gamma}{\partial\tau} \quad (11)$$

$$\Gamma(\tau=0)=0; \quad \Gamma(\xi=0)=\text{finite}; \quad \Gamma(\xi=1)=\Gamma_0$$

which are identical to Eq. (9) and its conditions. The solution for $\Gamma(\xi,\tau)$ can then easily be obtained following Example 19.1-2 of Bird, Stewart, and Lightfoot.

CONCLUSIONS

It has been shown how models of transient transport, when a boundary or driving-force change is introduced following an original steady-state operation, can be made to be mathematically identical in terms of the governing partial differential equation and initial and boundary conditions, to the respective constant-initial-condition models. This method should give transport teachers an alternative approach for not only setting up new unsteady-state classroom problems, but also for casting existing literature examples in a more general context.

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