

SCALING OF DIFFERENTIAL EQUATIONS

“Analysis of the Fourth Kind”

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What does it mean to solve a differential equation? The answer might be in closed form, or it can be an infinite series. A numerical simulation might also provide the answer. The first kind of answer is preferred but not always available or even possible. The second answer is useful if the series converges well, but this is not guaranteed in all cases. The third kind of answer is the least flexible, and doubt about the exactness of the simulation can remain.

This paper concerns a fourth kind of analysis, where a solution *per se* is not found, but the student learns about the dependence of the solution on relevant parameters and/or obtains an order of magnitude estimate of various meaningful quantities, such as the approximate thickness of a boundary layer. This answer is the result of natural scaling of the differential equation; it provides insight into an equation even when the solution to the equation or set of equations is unknown. This process of deducing relationships among the physical properties and significant dimensions of the problem accelerates physical understanding of its nature. The answers from this type of analysis often guide experiments, reducing their number to a minimum. Finally, the analysis can demonstrate that effects are important or unimportant.

The goal is to present an approach for arriving at the fourth kind of answer. The procedure is called “all-natural scaling” of the equation. There is at least one contribution in the literature on a similar topic. Hellums and Churchill^[1] described a general method for analyzing equations; their method reveals cases where similar solutions are found and at least indicates minimum numbers of parameters and variables. Their approach is formal and aimed more at deducing constraints on problems than on deducing physically meaningful quantities.

What need does this contribution fill? It is not a scientific advance, because scaling of equations has been around for a long time; scaled equations are the standard form in journal publications. For most undergraduates, the limited need for this understanding and the modest potential for comprehension of its significance are not compelling arguments for in-

roducing them to it. Likewise, this contribution is not intended for the experienced analyst who performs these operations subconsciously or has seen them all.

This method is intended primarily for advanced undergraduates or first-year graduate students who find themselves in classes where the professor conjures dimensionless groups without arguing their origins. I introduce this technique to the students in our core graduate math and transport courses; they seem not to have seen a direct discussion of this process before. This contribution is intended to fill that gap.

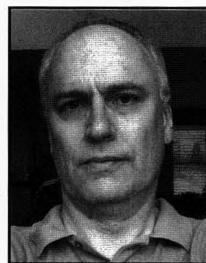
EXAMPLE 1

Viscous Heating and the Brinkman Number

Consider first the classic problem of viscous heating appearing in Figure 1. A warm viscous liquid flows laminarily in a pipe and is cooled by contact with the cold wall; the concern is whether or not viscous heating of the liquid is important. For simplicity, it is assumed that axial convection of energy dominates axial conduction, so that the important heat transfer terms are radial conduction, and viscous dissipation. The following equation governs convective heat transfer in laminar pipe flow under these circumstances:

$$\rho c_p v_z \frac{\partial}{\partial z} = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right] + \mu \left(\frac{\partial v_z}{\partial r} \right)^2 \quad (1)$$

where T = temperature, T_o = incoming temperature, T_w = wall



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temperature, v_z = axial velocity in laminar pipe flow, ρ = density of the fluid, μ = viscosity, c_p = heat capacity, k = thermal conductivity, r = radial position, and z = axial position.

Equation 1 is the convective conduction equation for the laminar flow of fluid in a pipe plus a term describing the local dissipation of mechanical energy into thermal energy.^[2] Before going to the trouble of solving the equation, or looking up the answer, we can use a scaling analysis to estimate the importance of the effect. This example illustrates the process of natural scaling and the deduction of the pertinent dimensionless group.

First, we pick all sensible length scales for the independent variables in the governing equation. R is obvious for radius, but there is no obvious choice for axial distance. We therefore temporarily give the axial length scale a name and deduce it during the derivation. This lets the equation exhibit appropriate relations among the physical properties. Finally, we define a dimensionless dependent variable preferably so that its value varies from zero to unity, when its range is known.

$$\xi \equiv \frac{r}{R} \quad \zeta \equiv \frac{z}{z_0} \quad \theta \equiv \frac{T - T_w}{T_0 - T_w} \quad (2)$$

For laminar pipe flow: $v_z = 2 \langle v \rangle (1 - \xi^2)$

Substitute these definitions into the equation using the chain rule for derivatives. The first crucial step is to divide by the coefficient of an important term in the equation. In this case, we are exploring the importance of the viscous heating term, so its coefficient must float. Axial convection of energy is obviously an important term, so one divides through the equation by the convective energy transport coefficient

$$2\rho c_p \langle v \rangle \left(\frac{T_0 - T_w}{z_0} \right) \quad (3)$$

The result is

$$(1 - \xi^2) \frac{\partial \theta}{\partial \zeta} = \frac{k z_0}{2 \rho c_p \langle v \rangle R^2} \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \theta}{\partial \xi} \right) \right] + \frac{16 \mu \langle v \rangle z_0}{2 \rho c_p R^2 (T_0 - T_w)} \xi^2 \quad (4)$$

Dividing the energy equation by Eq. (3) “scales” the axial convection term to 0(1); it declares axial convection to be important. The choice of which term to use in scaling the equation seems arbitrary at first. (Hellums and Churchill,^[1] for example, use the coefficient of the diffusive term to scale their Eqs. 10-12 but do not comment on the choice.) This

choice is not often critical as long as the term chosen is important in the problem. The first exercise of the Appendix of this contribution illustrates this point.

The radial conduction term is also important; after all, this is how the thermal energy escapes the pipe. Thus, the conduction term is scaled to 0(1) by equating its coefficient to unity and solving for the unknown length scale.

$$z_0 \equiv \frac{2 \langle v \rangle R^2 \rho c_p}{k} \quad (5)$$

With the inclusion of this axial length scale, the overall energy

equation can now be written as

$$(1 - \xi^2) \frac{\partial \theta}{\partial \zeta} = \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \theta}{\partial \xi} \right) \right] + 16 \text{Br} \xi^2 \quad (6)$$

where

$$\text{Br} \equiv \frac{\mu \langle v \rangle^2}{k(T_0 - T_w)} \quad (7)$$

The analysis yields two results. First, the temperature of the incoming fluid changes substantially toward the wall temperature over a distance z_0 that is calculable from known quantities of the problem. Second, the resulting parameter in Eq. 7, (Br), is a dimensionless group that governs the importance of viscous heating;^[2] *i.e.*, we can now quickly determine the significance of viscous heating relative to the ability of the system to dissipate the irreversible energy released. If the thermal conductivity is high relative to heating by viscous dissipation, the latter is unimportant. The effect of viscous heating is proportional to the viscosity and the square of the velocity, and inversely proportional to conductivity of the liquid. If 16Br is very small, we can ignore viscous heating—the usual case; otherwise, we should consult the published work.^[2]

Guidelines ■ The method used in the previous example consisted of several steps.

- 1) Write the governing equation including effects of interest.
- 2) Make position variables dimensionless with distances over which the dependent variable assumes the full range of its possible values. Where there is no obvious appropriate distance, give it a name and try to deduce it as part of the analysis (remember R and z_0).
- 3) Nondimensionalize the dependent variables with their full scale values.
- 4) Substitute the definitions into the differential equation using the chain rule for derivatives. Once students do this a couple of times, they easily write down the substituted form by inspection.
- 5) Identify a term of known importance and divide the equation by the coefficient of that term. This forces that term to order unity importance in the equation and scales the rest of the equation to that term. The equation becomes dimensionless.
- 6) Inspect the remaining terms of the equation. Whenever a co-

efficient contains only one unknown distance or other normalizing quantity and is also a known important term, set the coefficient to unity and solve for the unknown quantity (i.e., we knew the conduction in the radial direction was important, so we found z_0 with the coefficient of the conduction term.)

- 7) Collect remaining terms into as few coefficients as possible. These terms are generally dimensionless ratios that appear as parameters of the final solution.

These steps should be considered general guidelines. For the student, it is useful to try scaling the same equations by the coefficients of various terms to see the effect on the results. This process develops insight and experience that make the analysis meaningful. If one plans to solve the complete equation in closed form, the choice of reference distances does not matter. If we plan to solve the equation numerically, it can make a great deal of difference if the equation is properly scaled.

EXAMPLE 2

Natural Convection Near a Vertical Heated Surface

How much can be said about a classic case of natural convection without actually solving the governing equations in detail? Consider a heated vertical plate immersed in a fluid of infinite extent as shown in Figure 2. The well-known equations for the laminar case ($GrPr < 10^9$) are the following:

Continuity

$$\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad (8)$$

Motion

$$\rho \left(v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left(\frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g \beta (T - T_c) \quad (9)$$

Energy

$$\rho c_p \left(v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \quad (10)$$

where $v_y = y$ velocity, $v_z = z$ velocity, $T =$ temperature, $T_h =$ wall temperature, $T_c =$ bulk fluid temperature, $c_p =$ thermal heat capacity, $k =$ thermal conductivity, $g =$ gravity, $\beta =$ coefficient of expansion, $\rho =$ density, $\mu =$ viscosity, $y =$ horizontal position, and $z =$ vertical position.

For completeness, no assumption has been made about the relative importance of conduction or convection in the direction parallel to the wall. The first step is to identify scaling parameters for the independent variables, in this case y and z . The scaling distance for z is obviously H ; the scaling distance for y is unclear since the domain is infinite in that direction. Thus, define a distance y_0 as the appropriate scale for y . This distance is essentially a characteristic hydrodynamic boundary-layer thickness. Then define the dependent variable over its range

$$\zeta \equiv \frac{z}{H} \quad \eta \equiv \frac{y}{y_0} \quad \theta \equiv \frac{T - T_c}{T_h - T_c} \quad (11)$$

Likewise, there are no natural reference velocities for the vertical and horizontal velocities, so give them names as well ($\phi_z \equiv v_z / v_{oz}$, $\phi_y \equiv v_y / v_{oy}$) and define $B = \rho g \beta (T_h - T_c)$. After inserting them into the momentum equation, we obtain

$$\frac{\rho v_{oy} v_{oz}}{y_0} \left(\phi_y \frac{\partial \phi_z}{\partial \eta} \right) + \frac{\rho v_{oz}^2}{H} \left(\phi_z \frac{\partial \phi_z}{\partial \zeta} \right) = \frac{\mu v_{oz}}{y_0^2} \left(\frac{\partial^2 \phi_z}{\partial \eta^2} \right) + \frac{\mu v_{oz}}{H^2} \left(\frac{\partial^2 \phi_z}{\partial \zeta^2} \right) + B \theta \quad (12)$$

The convection of momentum in the direction parallel to the wall is surely important; scale the equation by dividing through by that term's coefficient

$$\frac{H v_{oy}}{y_0 v_{oz}} \left(\phi_y \frac{\partial \phi_z}{\partial \eta} \right) + \phi_z \frac{\partial \phi_z}{\partial \zeta} = \frac{v H}{y_0^2 v_{oz}} \left(\frac{\partial^2 \phi_z}{\partial \eta^2} \right) + \frac{v}{H v_{oz}} \left(\frac{\partial^2 \phi_z}{\partial \zeta^2} \right) + \frac{B H}{\rho v_{oz}^2} \theta \quad (13)$$

At this point, there are two terms that contain only one of the unknown reference variables—the second and third terms on the right-hand side. Typically, diffusion of momentum is negligible compared to convection of momentum in the primary direction of flow, thus it would not be prudent to base the definition of the reference velocity in the z -direction on the coefficient of this term. Furthermore, we know that for natural convection, the source term for momentum must be $O(1)$ or the problem does not make sense. Force the coefficient of this term to unity. We conclude that a reference velocity for the flow parallel to the vertical wall should be

$$v_{oz} \equiv \sqrt{\frac{B H}{\rho}} \quad (14)$$

Having this definition, we can now define other reference quantities by forcing the coefficients of other important terms to unity. The coefficient of the y -directed momentum diffusion terms yields

$$y_0 = \left(\frac{\mu^2 H}{\rho B} \right)^{1/4} \quad \text{and} \quad v_{oy} = \left(\frac{\mu^2 B}{\rho^3 H} \right)^{1/4} \quad (15)$$

and the differential equation becomes

$$\phi_y \frac{\partial \phi_z}{\partial \eta} + \phi_z \frac{\partial \phi_z}{\partial \zeta} = \frac{\partial^2 \phi_z}{\partial \eta^2} + \frac{\mu^2}{H^3 \rho B} \left(\frac{\partial^2 \phi_z}{\partial \zeta^2} \right) + \theta \quad (16)$$

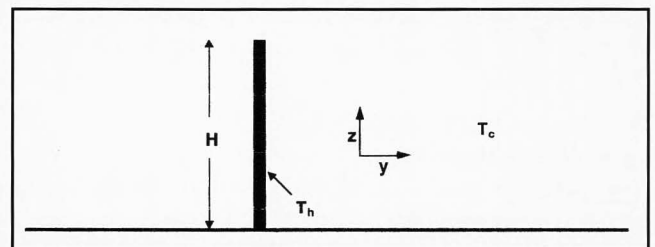


Figure 2. Geometry for natural convection near a heated wall.

This is as it should be. The typically important boundary-layer type terms are all of order unity along with the source term driving them. The axial diffusion of momentum is multiplied by a coefficient that allows its importance to be assessed. For even very modest temperature differences between the wall and the bulk fluid, or for large H , this term is small. The H^3 dependence of this parameter is very strong.

We now insert the definitions obtained into the energy equation and obtain

$$\left(\phi_y \frac{\partial \theta}{\partial \eta}\right) + \left(\phi_z \frac{\partial \theta}{\partial \zeta}\right) = \left(\frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2}\right) + \left(\frac{\alpha^2 \rho}{H^3 B}\right)^{1/2} \left(\frac{\partial^2 \theta}{\partial \zeta^2}\right) \quad (17)$$

The equation contains two parameters— Pr and a coefficient multiplying the axial diffusion term. Assuming that the axial diffusion of energy can be neglected, we find that the Prandtl number is the sole parameter of the system of Eqs.(8,9)

What happened to the Grashof number? Why does it not appear in this equation? To see how Gr arises, examine the flux of heat at the vertical wall, using the derived definitions to make it dimensionless

$$q \equiv h(T_w - T_c) = -k \frac{\partial T}{\partial y} \Big|_{y=0} \Rightarrow Nu = \frac{hy_o}{k} = \frac{h}{k} \left(\frac{\mu^2 H}{\rho B}\right)^{1/4} = -\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} \quad (18)$$

Still no Grashof number appears. Note that the appropriate scaling distance for heat flux normal to the wall is the hydrodynamic boundary-layer thickness y_o . The Nusselt number, *i.e.*, the dimensionless flux of heat, remains solely a function of Pr . The only way that Gr appears in the equation is if we convert this “all natural” scaling to one based on H as the length parameter. Then the flux equation becomes

$$q \equiv h(T_w - T_c) = -k \frac{\partial T}{\partial y} \Big|_{y=0} \Rightarrow Nu_H = Nu_{y_o} \frac{H}{y_o} = -\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} \left(\frac{\rho B}{\mu^2 H}\right)^{1/4} \frac{H}{1} \quad (19)$$

The coefficient on the far right-hand side is recognizable as Gr so that the definition of Nu_H becomes

$$Nu_H = -\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} Gr^{1/4} \quad (20)$$

The dimensionless temperature gradient at the wall is a function solely of the Pr number, as we found scaling of the system of coupled equations and is most often written as

$$-\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} \equiv f(Pr) Pr^{1/4} \quad (21)$$

where $f(Pr)$ is a slowly varying function of Pr . This definition leads to the tidy form

$$Nu_H = f(Pr)(Gr Pr)^{1/4} \quad (22)$$

which is the one commonly encountered.

As in the first example, there are several useful results. First, we now have estimates of the velocities achieved in the problem and the boundary layer thickness (Eqs. 14, 15). Second, we show that if axial diffusion of momentum and energy is small, the solution to the problem is only a function of Pr . Third, the origin of the Grashof number in this problem is clearly demonstrated.

CONCLUSIONS

Scaled equations are the standard for most journal publications, but apart from this standard, the process of scaling differential equations is a way to learn about their nature and build arguments about what terms can be neglected. The method requires that the student be able to read the equations at hand; in the examples, the student needs to recognize diffusive and convective terms. We suggest that this perspective be imparted concurrently with the method where necessary. We hope the method presented here helps advanced undergraduates and first-year graduate students become accustomed to the practice of scaling equations and, most of all, to understand the origin of dimensionless numbers, the shorthand of our profession.

APPENDIX: Suggested Further Examples

- 1) Repeat example 1, but divide through by the conductive term rather than the convective term; compare the results to Eq. 7.
- 2) One might object and say that it is strange to force all the terms to unity in example 2, that this must create an imbalance in the equation. We can check for suitability by inserting the definitions into the continuity equation. Problems with the scaling might appear there. Put the given definitions for the reference quantities into the continuity equation and deduce its form. Does a problem appear?
- 3) Consider the classic problem of flow of a free stream that meets and flows parallel to a flat plate. Include the axial diffusion of momentum. Deduce a parameter that allows one to estimate the minimum plate length for which axial diffusion of momentum can be neglected. Deduce an estimate of the thickness of the hydrodynamic boundary layer for a plate of length L . A close approximation to the exact answer is $5\sqrt{\nu L / v_\infty}$. How does your answer compare to this?
- 4) Write the energy equation for the above example, including the axial conduction term. Use the reference distances developed in Prob. 1. Deduce a parameter that allows estimation of the lengths below which axial conduction must be considered.
- 5) Instead of using the hydrodynamic boundary layer thickness in the energy equation, as in the previous problem, define a new reference length in the direction normal to the plate for the energy equation. Deduce an estimate of the thermal boundary layer thickness. Show that the ratio of the hydrodynamic layer thickness to the thermal layer thickness is given by $Pr^{1/2}$.

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2. Brinkman, H.C., *Appl. Sci. Research*, **A2**, p. 120, (1951).