ChE classroom

A New Approach to Teaching
TURBULENT THERMAL CONVECTION

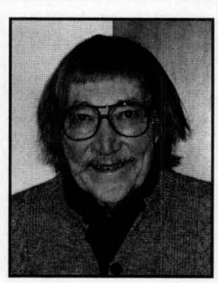
Stuart W. Churchill
University of Pennsylvania • Philadelphia, PA 19104-6393

At AIChE's annual meeting in 2000, I gave an oral presentation of an early version of a pair of new expressions, completely free of explicit empiricism, for the prediction of fully developed turbulent thermal convection in all channels and for all thermal boundary conditions. At the same venue, In 2001 I also presented a greatly improved version, although at the expense of a smidgen of empiricism. Both presentations prompted the same question from participants: "Is this approach being taught to current students, and if not, why not?" I explained in both instances that this material is very new and is not in any textbooks, and furthermore, that it may not appear in textbooks for some time to come since the authors of transport textbooks must first become aware of the concept and its results, and then be convinced of its educational (as well as predictive superiority) over the method they are currently teaching. Also, as Anderson[1] has noted, textbooks in chemical engineering seem to have a unique longevity, and the more successful of them are replaced or revised only after long intervals of time.

Undoubtedly with these textbook characteristics in mind, my mentor and departmental chairman, Donald L. Katz, long ago made the suggestion (which to a young assistant professor was virtually an order) that every year I replace at least 20% of the graduate transport course content by embracing new developments in the literature. Throughout my career, that suggestion led to my use of notes incorporating these new segments, together with using a book or books as a supplement rather than the other way around. I conclude, a full half-century later, that this process of annual supplementation and revision has, by virtue of the associated forced self-study and self-learning in the fields of my teaching, more than compensated me (and perhaps my students) for the efforts, and that it is a worthy complement of the new materials most of us introduce periodically from our own research and consulting. I am here taking advantage of the platform provided by *Chemical Engineering Education* to encourage and assist the process of supplementation for transport teachers with respect to a new approach for the description and prediction of turbulent thermal convection.

In a previous *CEE* article,[2] I presented a new approach to the description and teaching of **turbulent flow** with the same objective. For that simpler and more restricted topic, it was possible to include in the presentation a virtually complete set of supplementary notes for direct use by any interested faculty member. For the much more complex process of turbulent thermal convection and the much more complex process of development of the new model, however, the presentation of a working set of supplemental notes in this format is simply not feasible. Rather, this article has the more limited objective of outlining the new approach with the hope that faculty members who teach transport will be inspired to study the more complete documentation in the key references and make the effort to formulate their own supplemental notes. Perhaps I will eventually find the time and motivation to prepare a monograph on this topic, but I do not recommend that anyone procrastinate with that as the excuse.

When an analogue of the approach that was so simple,



__Stuart W. Churchill__ is the Carl V.S. Patterson Professor Emeritus at the University of Pennsylvania, where he has been since 1967. His BSE degrees (in ChE and Math), MSE, and PhD were all obtained at the University of Michigan, where he also taught from 1950 to 1967. Since his formal retirement in 1990, he has continued to teach and carry out research on heat transfer and combustion. He is also currently completing books on turbulent flow and correlation.

straightforward, and successful for turbulent flow was first attempted for the closely related topic of turbulent thermal convection, I anticipated that the path of development would closely parallel the previous one. While convection is inherently more complex than flow in several respects, it is also simpler in the sense that it merely consists of the superposition of a scalar quantity, the temperature, on the flow. The path of development that emerged after considerable trial and error proved to reflect the greater complexity that had been anticipated, and the final results proved to reflect the anticipated greater simplicity.

The predictive equations for turbulent thermal convection that are described in this paper are, by a significant margin, more accurate, fundamentally sound, and general than any prior ones. They also provide better insight into the relationship between flow and convection and a better conception of thermal convection itself that more than compensates for the greater detail. This new material should therefore, as suggested by audience members at the AIChE presentations, be given serious consideration for inclusion in the final portfolios of both our undergraduate and graduate students.

Apart from the merit of the predictive equations for turbulent thermal convection that emerged, the path of their development appears to have merit itself in an educational sense. On the one hand, it provides insight into a creative process of correlation that is within the capabilities of our students. On the other hand, it provides a perspective within which the strengths and weaknesses of all forms of correlation can be evaluated, not only in flow and convection but also in every aspect of chemical engineering. Our students should be made to realize that whatever career they follow after graduation, they will spend considerable time using and/or formulating correlations.

I have a predilection for presentations in narrative and historical contexts under the presumption that the personal characteristics, as well as the triumphs and failures, of our predecessors not only stimulate interest but also provide a mnemonic for students. In this instance, a description of the serendipitous and irregular path of development of a completely new formulation in a relatively mature field may serve a similar role. Teachers who prefer a more orderly and skeletal approach are welcome to eliminate such diversionary material.

Many details concerning origins, proofs, uncertainties, and limitations are deferred to the references, and in particular to Churchill and Zajic.[3] It is, however, essential that the teacher present these details, or perhaps in the instance of graduate students, assign key references as required collateral reading. In either event, students should be encouraged to question the validity of the many assertions and simplifications in this article rather than accept them "on faith." Undergraduate students may require more guidance than do graduate students with respect to the new approach, but they have the counterbalancing advantage of less to unlearn.

THE NEW APPROACH FOR TURBULENT FLOW

A thorough understanding by students and faculty alike of the new approach for the description and teaching of turbulent flow, as previously described[2], is an essential prerequisite for the complementary new approach presented here for turbulent thermal convection. Because of space limitations, however, only those results that are directly applied or adapted for thermal convection will be reproduced here.

The time-averaged, once-integrated differential equation of conservation for momentum in the radial (negative-y) direction in steady-on-the-mean, full developed flow of a fluid of invariant density and viscosity through a round tube can be represented by

$$\tau_w\left(1-\frac{y}{a}\right)=\mu\frac{du}{dy}-\rho\overline{u'v'} \tag{1}$$

Here, τ_w is the shear stress on the wall, y is the distance from the wall, a is the radius of the pipe, u is the time-averaged velocity, and u' and v' are the fluctuating components of the velocity in the x and y directions, respectively. The superbar designates the time-average of their product, while μ and ρ are the dynamic viscosity and specific density of the fluid. (*Aside to teachers*: The origin of this expression and the physical meaning of the several variables and terms, including the signs of the latter, should be described or reviewed as appropriate. Any uneasiness of the students in this regard can be expected to persist in what follows. Of course, this warning applies to some extent to subsequent details as well.)

Equation (1) can be rewritten in terms of the dimensionless "wall" variables of Prandtl, namely

$$u^+ = u(\rho/\tau_w)^{1/2}$$

$$y^+ = y(\tau_w\rho)^{1/2}/\mu$$

$$a^+ = a(\tau_w\rho)^{1/2}/\mu$$

and one new variable, namely the fraction of the transport of momentum (or the total shear stress) due to the turbulent fluctuations $\left(\overline{u'v'}\right)^{++} = -\rho\overline{u'v'}/\tau$ as

$$\left(1-\frac{y^+}{a^+}\right)\left[1-\overline{u'v'}^{++}\right]=\frac{du^+}{dy^+} \tag{2}$$

Equation (1), with y^+/a^+ replaced by 1-R, can be integrated formally to obtain the following expression for the radial distribution of the time-averaged velocity:

$$u^+ = \frac{a^+}{2}\int_{R^2}^{1}\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^2 \tag{3}$$

The velocity distribution can in turn be integrated over the cross-section to obtain, after utilizing integration by parts, the following integral expression for the mixed-mean velocity and thereby the Fanning friction factor:

$$\left(\frac{2}{f}\right)^{1/2} = u_m^+ \equiv \int_0^1 u^+ dR^2 = \frac{a^+}{4}\int_0^1 \left[1-\left(\overline{u'v'}\right)^{++}\right] dR^4 \qquad (4)$$

Equations (1) through (4) are exact insofar as the restrictions mentioned above with respect to Eq. (1) are fulfilled. In order to implement Eqs. (3) and (4), an expression is required for $\left(\overline{u'v'}\right)^{++}$ in terms of y^+ and a^+. For this purpose, Churchill[4] proposed the following semi-empirical expression:

$$\left[\left(\overline{u'v'}\right)^{++}\right]^{-8/7} = \left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7} + \left|\exp\left\{\frac{-1}{0.436y^+}\right\} - \frac{1}{0.436a^+}\left(1+\frac{6.95y^+}{a^+}\right)\right|^{-8/7} \qquad (5)$$

It is essential for the students to be aware of the origins and uncertainties of Eq. (5) since this expression has a critical role, both numerically and functionally, in all of the developments that follow for both flow and convection. The third-power dependence on y^+ for small values of y^+ was originally postulated on the basis of asymptotic analyses, but has since been confirmed by *direct numerical simulations,* which have also produced a theoretical value of approximately 7×10^{-4} for the numerical coefficient. The exponential term for moderate values of y^+, as well as the deductive term for $y \to a^+$ were both derived by speculative analysis, but the coefficients of 0.436 and 6.95 were determined from recent, improved experimental data for the time-averaged velocity distribution. The power-mean form of Eq. (5) is arbitrary and the combining exponent of -8/7 is based on experimental data for $\overline{u'v'}$. (See Churchill and Zajic[3] for further details, including complete references.)

Numerical integration of Eqs. (3) and (4) using $\left(\overline{u'v'}\right)^{++}$ from Eq. (5) results in almost exact values of u^+ and u_m^+ owing to the smoothing associated with integration. Such values of u_m^+ may be represented with a high degree of accuracy for $a^+ > 300$ by the following expression that invokes no additional empiricism beyond that of Eq. (5):

$$\left(\frac{2}{f}\right)^{1/2} = u_m^+ = 3.2 - \frac{227}{a^+} + \left(\frac{50}{a^+}\right)^2 + \frac{1}{0.436}\ell n\{a^+\} \qquad (6)$$

Equations (1) through (6) are the only ones for flow that will be referred to directly in the developments that follow for convection.

It may occur to teachers and graduate students at this point that the relevant consideration of turbulent flow has been completed without any mention of the eddy viscosity or the mixing length. One merit of the new approach, which carries over to thermal convection, is that the need to introduce such heuristic quantities is avoided completely by the more direct and simple development in terms of $\left(\overline{u'v'}\right)^{++}$.

AN ASIDE ON A GENERIC CORRELATING EQUATION

Equation (5) is a particular application of the generic correlating equation proposed by Churchill and Usagi[5] for two regions, namely

$$y^b = y_0^b + y_\infty^b \qquad (7)$$

Here, $y = y\{x\}$, $y_0 = \{x \to 0\}$, $y_\infty = y\{x \to \infty\}$, and b is an arbitrary exponent. Either y_0 or y_∞ or both are necessarily functions of x rather than fixed values. For three regions, Eq. (7) can be extended either directly as

$$y^{bq} = \left(y_0^b + y_i^b\right)^q + y_\infty^{bq} \qquad (8)$$

or in staggered form as

$$\left(y^b - y_0^b\right)^q = y_i^{bq} + \left(y_\infty^b - y_0^b\right)^q \qquad (9)$$

Here, y_i is an intermediate asymptote and q is a second arbitrary exponent. The reverse order of combination of y_0, y_i, and y_∞ leads to equally valid and, in general, fundamentally different representations. Equations (7) through (9) have been introduced here to avoid interrupting the continuity of the development in which they are used.

DEVELOPMENT OF A NEW FORMULATION FOR TURBULENT CONVECTION

The analogue of Eq. (1), with the additional idealization of negligible viscous dissipation, is

$$j = -k\frac{\partial T}{\partial y} + \rho c\overline{T'v'} \qquad (10)$$

and that of Eq. (2) is

$$\frac{j}{j_w}\left[1-\left(\overline{T'v'}\right)^{++}\right] = \frac{\partial T^+}{\partial y^+} \qquad (11)$$

Here, j is the heat flux density in the y-direction, T is the temperature of the fluid, j_w and T_w are their values at the wall, $T^+ = k(\tau_w\rho)^{1/2}(T_w - T)/\mu j_w$, $\overline{T'v'}$ is the time-averaged product of these fluctuating quantities, $\left(\overline{T'v'}\right)^{++} = \rho c\overline{T'v'}/j$ is the fraction of the radial heat flux density due to the turbulent fluctuations, and k is the thermal conductivity of the fluid. The terms j/j_w and $\left(\overline{T'v'}\right)^{++}$ in Eq. (11) depend on two parameters, namely the Prandtl number $Pr = c\mu/k$ and the mode of heating at the wall, as well as on y^+ and a^+.

From an energy balance over an inner cylindrical segment of the fluid stream, it follows that

$$\frac{j}{j_w}=\frac{1}{R}\int_0^{R^2}\frac{u}{u_m}\left(\frac{\partial T/\partial x}{\partial T_m/\partial x}\right)dR^2 \tag{12}$$

Here, T_m is the mixed-mean temperature of the fluid stream. As contrasted with τ/τ_w, which may be inferred from Eq. (1) to vary linearly with R, j/j_w varies non-linearly because of its dependence on the velocity distribution and in some instances on the temperature distribution as well. Also, as can be inferred from Eq. (12), T varies with x as well as with y, even in fully developed thermal convection, whereas u varies only with y in fully developed flow. Fully developed thermal convection is ordinarily defined by two criteria, namely

$$\frac{\partial}{\partial x}\left(\frac{T_w-T}{T_w-T_m}\right)\equiv 0 \quad \text{and} \quad \frac{\partial h}{\partial x}\equiv 0$$

where $h = j_w/(T_w - T_m)$ is the local heat transfer coefficient.

Equation (11) can be put in a more tractable form for both formal and numerical solution by introducing new variables γ and Pr_t defined as follows, in place of j/j_w and $\left(\overline{T'v'}\right)^{++}$

$$1+\gamma=\frac{j}{j_w}\left(\frac{\tau_w}{\tau}\right)=\frac{j}{j_w R}=\frac{1}{R^2}\int_0^{R^2}\frac{u}{u_m}\left(\frac{\partial T/\partial x}{\partial T_m/\partial x}\right)dR^2 \tag{13}$$

and

$$\frac{Pr_t}{Pr}\equiv Pr\left(\frac{1-\left(\overline{T'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)\frac{\left(\overline{u'v'}\right)^{++}}{\left(\overline{T'v'}\right)^{++}} \tag{14}$$

The result is

$$\frac{(1+\gamma)R}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{\left(\overline{u'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)}=\frac{dT^+}{dy^+} \tag{15}$$

The use of γ, the perturbation of the heat flux density distribution from that of the shear stress distribution, was suggested by Reichardt.[6] The variable Pr_t was originally introduced in connection with modeling in terms of the eddy viscosity and eddy conductivity, and accordingly, by analogy with the corresponding ratio of molecular quantities, was called the turbulent Prandtl number. Although the redefinition of Pr_t in terms of $\left(\overline{u'v'}\right)^{++}$ and $\left(\overline{T'v'}\right)^{++}$ avoids these heuristic variables, the traditional name and symbol for this quantity are retained herein out of respect for its historical origin. It should be noted that Pr_t is not necessarily proportional to Pr since $\left(\overline{T'v'}\right)^{++}$ is, in general, a function of Pr.

Equation -=R can be integrated formally to obtain

$$T^+=\frac{a^+}{2}\int_{R^2}^{1}\frac{(1+\gamma)dR^2}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{\left(\overline{u'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)} \tag{16}$$

Then T^+, weighted by u^+/u_m^+, can be integrated over the cross section to obtain

$$Nu\equiv\frac{2a^+}{Tm^+}=4\Bigg/\int_0^1\left[\int_{R^2}^{1}\frac{(1+\gamma)dR^2}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{\left(\overline{u'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)}\right]\left(\frac{u}{u_m}\right)dR^2 \tag{17}$$

For uniform heating at the wall, it follows from the criteria for fully developed thermal convection that $\partial T/\partial x=\partial T_m/\partial x$. It then follows from the correspondingly reduced form of Eq. (13), together with Eqs. (3) and (4), that γ is a function only of y^+ and a^+. Equation (17) can then by virtue of the same considerations, be integrated by parts to obtain

$$Nu=8\Bigg/\int_0^1\frac{(1+\gamma)^2dR^4}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{\left(\overline{u'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)} \tag{18}$$

Equation (18) can be reduced for three special cases. For Pr = 0, it can be expressed as

$$Nu_0\equiv Nu\{Pr=0\}=8\Big/\int_0^1(1+\gamma)^2dR^4=8\big/(1+\gamma)^2_{mR^4} \tag{19}$$

while for Pr_t = Pr, it can be reduced by virtue of Eq. (4) to

$$Nu_1=Nu\{Pr=Pr_t\}=\frac{8}{\int_0^1(1+\gamma)^2\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^4}=$$

$$\frac{8}{\int_0^1\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^4}\left(\frac{\int_0^1\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^4}{\int_0^1\left[1-\left(\overline{u'v'}\right)^{++}\right](1+\gamma)^2dR^4}\right)=$$

$$\frac{2a^+}{u_m^+(1+\gamma)^2_{wmR^4}} \tag{20}$$

Here, as can be inferred, $(1+\gamma)^2_{mR^4}$ designates the integrated-mean value over R^4, and $(1+\gamma)^2_{wmR^4}$ the integrated-mean value weighted by 1 - $\left(\overline{u'v'}\right)^{++}$. Both quantities may readily be evaluated numerically, using Eqs. (3), (4), and (5), and the reduced form of Eq. (13). For Pr $\rightarrow\infty$, the temperature field develops almost completely very near the wall where $\left(\overline{u'v'}\right)^{++}$ can be approximated by 0.7 $(y^+/10)^3$ and γ can be neglected. Equation (16) can then be integrated in closed form to obtain

$$Nu_\infty = Nu\{Pr \rightarrow \infty\} = 3^{3/2}(0.0007)^{1/3} a^+ (Pr/Pr_t)^{1/3} / \pi = 0.07343 Re(f/2)^{1/2} (Pr/Pr_t)^{1/3} \quad (21)$$

For uniform wall temperature, the criteria for fully developed convection require that

$$(\partial T/\partial x)/(\partial T_m/\partial x) \cong T^+ / T_m^+$$

Integration of Eq. (17) by parts is no longer possible, but from the limiting form of Eq. (16) for R = 0, it follows that

$$Nu_0 = 4\left(T_c^+ / T_m^+\right)_0 \Big/ (1+\gamma)_{mR^2} \quad (22)$$

and

$$Nu_1 = 4\frac{u_m^+}{u_c^+}\left(\frac{T_c^+}{T_m^+}\right)_1 \frac{Re\left(f/2\right)}{(1+\gamma)_{wmR^2}} \quad (23)$$

Here, T_c is the temperature at the axis of the pipe. Equation (21) remains applicable as is. The determination of numerical values of γ, T_c^+, and T_m^+ from Eqs. (13), (16), and (17) now requires iteration, but the functional forms of Eqs. (22) and (23) are adequate for the development herein.

On the basis of the previous experiences with various aspects of turbulent flow, I anticipated that Eqs. (19) through (23) could be combined in appropriate pairings in the form of Eq. (7) to construct satisfactory correlating equations for $Pr \geq Pr_t$ and for $Pr \leq Pr_t$, or alternatively, in appropriate triplets in the form of Eq. (8). All such attempts failed, however. I then found (somewhat serendipitously) that a successful correlating equation for turbulent thermal convection could be devised by using a particular analogy between momentum and energy transfer in which the exact solutions for three particular values of Pr occur in the form of Eq. (9). Accordingly, a brief and very selective review of such analogies is appropriate at this point.

SELECTIVE ANALOGIES

Reynolds[7] postulated that the transport of both momentum and energy between a turbulent stream and its confining surface occurred wholly by means of a mass flux of eddies and thereby derived the equivalent of

$$Nu = Pr\, Re(f/2) \quad (24)$$

Prandtl[8] improved upon the Reynolds analogy by postulating an added resistance due to linear molecular diffusion of momentum and energy across a viscous boundary layer of thickness δ in series with transport by the eddies of Reynolds in the turbulent core, thereby obtaining the equivalent of

$$Nu = \frac{Pr\, Re(f/2)}{1 + \delta^+(Pr-1)(f/2)^{1/2}} \quad (25)$$

Equation (25), just as Eq. (24), is inapplicable for Pr < 1, owing to neglect of thermal conduction in the turbulent core, and also for Pr >> 1, owing to neglect of eddy transport within the viscous boundary layer. Even so, it represents a great advance in that it correctly predicts a coupled, non-power dependence on both Pr and Re, in the latter case by virtue of the dependence of f on Re. Of the many analogies that have been proposed to eliminate the deficiencies of the Prandtl analogy for large and small values of Pr (see, for example, Churchill[9]), only two need to be examined here.

Reichardt[6] eliminated dy^+ between the equivalents of Eqs. (2) and (15) and made several ingenious approximations that allowed him to integrate the resulting combined equation in closed form. Churchill[9] assembled the fragments of this solution into a single expression for Nu and corrected the erroneous expression used by Reichardt for the shear stress near the wall, thereby obtaining

$$\frac{1}{Nu} = \frac{(1+\gamma)_{mu^+}}{Re(f/2)}\left(\frac{T_m^+}{T_c^+}\right)\left(\frac{u_c^+}{u_m^+}\right)\left(\frac{Pr_t}{Pr}\right) + \frac{13.62}{Re(f/2)^{1/2}}\left(\frac{T_m^+}{T_c^+}\right)\left(1-\frac{Pr_t}{Pr}\right)\left(\frac{Pr_t}{Pr}\right)^{1/3} \quad (26)$$

Equation (26) is limited in applicability to $Pr \geq Pr_t$ by virtue of one of the simplifications made by Reichardt in order to be able to integrate analytically.

Churchill[10] (also Churchill and Zajic[3]) followed a completely different path to derive an expression, which for $Pr \geq Pr_t$, is exactly equivalent to Eq. (26) except for replacement of the term $1 - Pr_t/Pr$ by $1 - (Pr_t/Pr)^{2/3}$. In retrospect, the difference in these expressions is a consequence of the approximation of Reichardt of du^+ by dy^+ in the differential term leading to the right-most term of Eq. (26).

FINAL FORMS

The final predictive expressions for turbulent thermal convection emerged from the various expressions above by means of the following lengthy series of insights, postulates, and inferences, all of which were essential.

(1) Churchill, *et al.*,[11] recognized that Eq. (26) was equivalent, with T_m^+ / T_c^+ evaluated at the limiting conditions, to

$$\frac{1}{Nu} = \left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left(1 - \frac{Pr_t}{Pr}\right)\frac{1}{Nu_\infty} \quad (27)$$

(2) They further recognized that when Eq. (17) was rearranged as

$$\frac{Nu - Nu_1}{Nu_\infty - Nu_1} = 1 \bigg/ \left[1 + \frac{Nu_\infty}{Nu_1}\left(\frac{Pr_t}{Pr - Pr_t}\right)\right] \quad (28)$$

it had the form of Eq. (9), with

$$b = -q = 1$$

$$y_0 = Nu_1$$

$$y_\infty = Nu_\infty$$

$$y_i = \frac{Nu_1}{Nu_\infty}(Nu_\infty - Nu_1)\left(\frac{Pr}{Pr_t} - 1\right)$$

The staggered independent variable, Pr/Pr_t - 1, has the essential role of converting Nu_1 from a particular value to an asymptote. According to Eq. (28), Nu goes through a sigmoidal transition from Nu_1 to N_∞, a nuance of behavior that had previously been overlooked. In retrospect, correlation in terms of Eq. (7), that is, direct interpolation between Nu_1 and Nu_∞, was doomed to fail. The relationship provided by the Reichardt analogy was essential to the derivation of Eq. (27).

(3) The identification of Eq. (28) with Eq. (9) suggested that the analogue of Eq. (28) in terms of Nu_0 and Nu_1 might be applicable for $Pr \leq Pr_t$. That concept led to an expression with a discrete step in the derivative of Nu with respect to Pr/Pr_t at $Pr = Pr_t$, but elimination of this discontinuity by means of an arbitrary but ultimately vanishing coefficient resulted in

$$\frac{Nu - Nu_0}{Nu_1 - Nu_0} = 1 \bigg/ \left[1 + \frac{Nu_1}{Nu_\infty^1}\left(\frac{Nu_\infty^1 - Nu_1}{Nu_1 - Nu_0}\right)\left(\frac{Pr_t - Pr}{Pr}\right)\right] \quad (29)$$

where $Nu_\infty^1 = Nu_\infty\{Pr = Pr_t\} = 0.07343\,Re(f/2)^{1/2}$.

(4) The absence of any allusion to geometry or to the thermal boundary condition suggested that Eqs. (28) and (29) might be applicable for all geometries and all thermal boundary conditions. Plots of numerically computed values of Nu versus Pr/Pr_t for round tubes with uniform heating and uniform wall temperature, and for parallel-plate channels with equal uniform heating and with unequal uniform temperatures, confirmed the validity of this speculation.

(5) These plots in logarithmic coordinates appeared to provide an excellent overall representation for all values of Pr/Pr_t, for all values of a^+ or b^+ (where b is the half-spacing of the parallel plates) greater than 145, which is the lower limit for the existence of fully turbulent flow, for all geometries, and for all thermal boundary conditions. The more critical test provided by arithmetic plots, however, reveal errors in Nu of up to 20% for both $Pr/Pr_t = 0\{10\}$ and $Pr/Pr_t = 0\{0.01\}$. After many attempted correctives, substitution of the analogy of Churchill for that of Reichardt to obtain

$$\frac{1}{Nu} = \left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left[1 - \left(\frac{Pr_t}{Pr}\right)^{2/3}\right]\frac{1}{Nu_\infty} \quad (30)$$

was found to result in an almost perfect representation for the dependence of Nu on Pr/Pr_t.

(6) The analogue of Eq. (30) for $Pr \leq Pr_t$, corrected as was Eq. (29) to remove the singularity in the derivative, and with the arbitrary inclusion of the empirical factor $(Pr_t/Pr)^{1/8}$, is

$$\frac{Nu - Nu_0}{Nu_1 - Nu_0} = 1 \bigg/ \left[1 + \frac{\left(\frac{Pr_t}{Pr} - 1\right)\left(Nu_\infty^1 - \frac{2}{3}Nu_1\right)Nu_1}{(Pr_t/Pr)^{1/8}(Nu_1 - Nu_0)Nu_\infty^1}\right] \quad (31)$$

This expression results in almost exact representations for $Pr < Pr_t$ for all of the previously mentioned conditions—thereby it is a complement in every respect to Eq. (30).

IMPLEMENTATION

The numerical calculation of values of Nu for specified values of Re and Pr and for particular geometries and boundary conditions requires numerical values or expressions for f, Nu_0, Nu_1, and Pr_t. For a round tube, values of f of sufficient accuracy can be determined from Eq. (6) by noting that $Re = 2a^+u_m^+$. Values of Nu_0 and Nu_1 can be calculated from Eqs. (19) and (20), but an array of such values has already been calculated for representative values of a^+, and correlating equations have been devised for interpolation. The slight inaccuracy associated with Eq. (5) is totally negligible when it is used in conjunction with Eqs. (19) and (20). Equivalent expressions for f, and values and expressions for Nu_0 and Nu_1 are also available or can readily be derived and calculated for other geometries and thermal boundary conditions. Equation (21) is directly applicable as an asymptote for large values of Pr for all geometries and conditions. Current correlative and predictive equations for Pr_t are quite uncertain (see, for example, Kays[12] or Churchill[13]). However, Nu as predicted by Eqs. (30) and (31) is fortuitously insensitive to the expression used for Pr_t, and the following purely empirical equation

$$Pr_t = 0.85 + \frac{0.015}{Pr} \quad (32)$$

appears to be adequate for that purpose. The dividing value of Pr with respect to the use of Eq. (30) or (31), that is, the value of Pr for which $Pr = Pr_t$, is 0.867 according to Eq. (32). Other correlating equations for Pr_t give only slightly different numerical values for this pivotal value of Pr. Either Eq. (30) or Eq. (31) can be used without serious error for $0.45 < Pr < 1.7$, which suggests that Eq. (30) is a sufficient expression for all fluids other than liquid metals.

SUMMARY

Equations (30) and (31), together with Eq. (32), predict values of Nu within 1% or 2% of numerically calculated values for all geometries and conditions in the fully turbulent regime. This is to be compared with deviations of 10% to 40% on the mean for all expressions in current use, many of which are greatly restricted with respect to range and conditions (see Churchill and Zajic[3]). The remarkable improvement in accuracy for $Pr \geq Pr_t$, as provided by Eq. (27), is a consequence of using the Reichardt analogy, which is free of any explicit empiricism. This expression fails in exactness only due to some minor mathematical simplifications made in its derivation. This slight inaccuracy is in turn virtually eliminated by use of the analogy of Churchill. On the other hand, the greatly improved accuracy of Eq. (31) for $Pr \leq Pr_t$ is a consequence of the identification of the structure of the analogy of Reichardt with that of the generic correlating equation of Churchill and Usagi for three regimes in staggered form, together with a minor empiricism. This same identification revealed a virtual regime and a point of inflection for $Pr \leq Pr_t$, and another such pair that had never before been recognized for $Pr > Pr_t$. The existence of these virtual regimes explains the numerical and functional failures of most prior correlating equations.

The generality of the new expressions for all geometries and thermal boundary conditions is a consequence of the recognition that the analogy of Reichardt could be expressed in terms of Nu_0, Nu_1, Nu_∞, and Pr/Pr_t. The supplementary expressions for Nu_0, Nu_1, and Nu_∞, which are exact insofar as Pr_t is independent of y^+, follow directly from formulation of the equations of conservation in terms of the fraction of the transport due to the turbulent fluctuations. They could have been derived using eddy diffusional models, but not so simply.

Implementation of the new expressions for specified values of Re and Pr, and for particular geometries and thermal boundary conditions, is not onerous since the entire calculation can be preprogrammed.

The path of development leading to Eqs. (30) and (31) could now be streamlined, but the description of the irregular path that was actually followed has educational value in that all students and practicing engineers should be concerned with the evaluation if not the construction of correlating equations.

Although the process of derivation of the new relationships for thermal convection is much more complicated, and the relationships themselves are slightly more complicated to employ, these deficiencies appear to be a small price to pay for their greater accuracy, sounder rationale, and broader applicability.

Students should be prompted to question any of the assertions and non-obvious steps that were made in the abbreviated development herein and not expanded upon by the teacher. Justifications may generally be found in the references.

REFERENCES

1. Anderson, T.J., "Chemical Processing of Electrons and Holes," *Chem. Eng. Ed.*, **24**(1), 26 (1990)
2. Churchill, S.W., "A New Approach to Teaching Turbulent Flow," *Chem. Eng. Ed.*, **32**(2), 142 (1999)
3. Churchill, S.W., and S.C. Zajic, "The Prediction of Turbulent Convection with Minimal Explicit Empiricism," *AIChE J.*, **48,** 927 (2002)
4. Churchill, S.W., "New Simplified Models and Formulations for Turbulent Flow and Convection," *AIChE J.*, **42**, 1125 (1997)
5. Churchill, S.W., and R. Usagi, "A General Expression for the Correlation of Rates of Transfer and Other Phenomena," *AIChE J.*, **18**, 1121 (1972)
6. Reichardt, H., "Die Grundlagen des Turbulenten Wärmeübertraganges," *Archiv ges. Wärmetechn.*, **2**, 129 (1951): English translation, "The Principles of Turbulent Transfer," *Nat. Advisory Comm. Aeronaut.*, TM 1408, Washington, DC (1957)
7. Reynolds, O., "On the Extent and Action of the Heating Surface of Steam Boilers," *Proc. Lit. Soc., Manchester,* **14**, 7 (1874)
8. Prandtl, L., "Ein Beziehung zwischen Wärmeaustaush und Strömungswiderstand der Flüssigkeiten," *Phys. Z.*, **11**, 1072 (1910)
9. Churchill, S.W., "Critique of the Classical Algebraic Analogies between Heat, Mass, and Momentum Transfer," *Ind. Eng. Chem. Res.*, **36**, 3878 (1987)
10. Churchill, S.W., "New Wine in New Bottles: Unexpected Findings in Heat Transfer. Part III. The Prediction of Turbulent Convection with Minimal Explicit Empiricism," *Thermal Sci. Eng.*, **5**(3), 13 (1997)
11. Churchill, S.W., M. Shinoda, and N. Arai, "A New Concept of Correlation for Turbulent Convection," *Thermal Sci. Eng.*, **8**(4), 49 (2000)
12. Kays, W.M., "Turbulent Prandtl Number: Where are We?" *J. Heat Transfer, Trans ASME*, **116**, 234 (1994)
13. Churchill, S.W., "A Reinterpretation of the Turbulent Prandtl Number," *Ind. Eng. Chem. Res.*, in press ❐

ChE letter to the editor

Dear Editor:

Late last year, you published our Letter to the Editor regarding a survey we were carrying out on the use of Inherently Safer Design (ISD), meant to make the process industry a lot safer. Several of your readers downloaded our questionnaire and sent their responses to us. We got responses from eleven countries world wide.

The findings of the survey have just been published under the title "Inherently Safer Design: Present and Future" in the *Transactions of the Institution of Chemical Engineers, Process Safety and Environmental Progress,* **80**, Part B, May 2002.

We are pleased to enclose a copy of the publication for