The object of this column is to enhance our readers' collections of interesting and novel problems in chemical engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class, or in a new light, or that can be assigned as a novel home problem, are requested, as well as those that are more traditional in nature and that elucidate difficult concepts. Manuscripts should not exceed 14 double-spaced pages and should be accompanied by the originals of any figures or photographs. Please submit them to Professor James O. Wilkes (e-mail: wilkes@umich.edu), Chemical Engineering Department, University of Michigan, Ann Arbor, MI 48109-2136.

SCALED SKETCHES FOR VISUALIZING SURFACE TENSION

SARAH L. MASON
Technical University of Denmark • Lyngby, Denmark

This article suggests using scaled sketches as a supplement to the usual tabular or graphical output of spreadsheet programs, and gives an example of using such sketches to visualize the action of surface tension. Many students find visual information such as pictures and diagrams useful,[3] and sketches are commonly used when presenting the concept of surface tension. While unscaled sketches are helpful, scaled sketches can more accurately illustrate quantitative relationships.

Scaled sketches can be drawn in an Excel spreadsheet environment and automated using the Visual Basic macro capability. These scaled sketches for visualizing surface tension are generated based on values of surface tension, radius, chord length, and internal pressure chosen by the student. The student can then view how the components of the tension and pressure forces balance when the parameter values change in the Laplace-Young equation.

Surface tension may be thought of both as an energy per unit area and as a force per unit length.[2] Using the force-balance approach allows sketches to be drawn and comparisons made among various situations where forces act on curved surfaces, such as for liquid drops, thin-walled pressure vessels, and even tensor bandages.

Sarah L. Mason *is associate professor of industrial rheology at BioCentrum-DTU at the Technical University of Denmark, near Copenhagen. She has degrees in chemical engineering from Queen's University and the University of Waterloo in Ontario, Canada, and the Swiss Federal Institute of Technology in Zurich, Switzerland (Ph.D. 1992). Research interests include the physics of emulsion formation and the rheology of structured food materials.*

2-D FORCE BALANCE

If we ignore the force of gravity, we can assume that the pressure is the same everywhere inside an object. In large liquid drops this is not strictly true and leads to flattened drops in reality. Nevertheless, ignoring gravity allows us to choose one radius of curvature for an object and write the Laplace-Young equation in two dimensions as

$$P_{In} = P_{Out} + \frac{\sigma}{R} \tag{1}$$

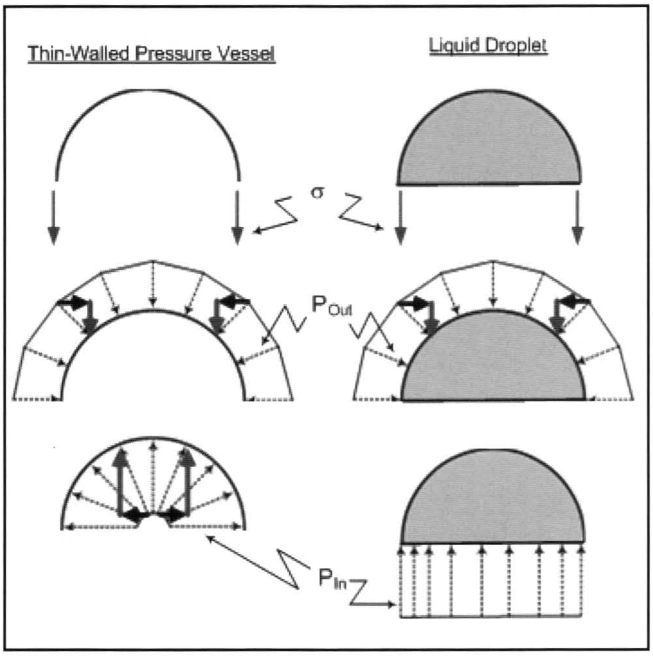


***Figure 1.** Forces applied to a thin-walled pressure vessel or a small liquid drop.*

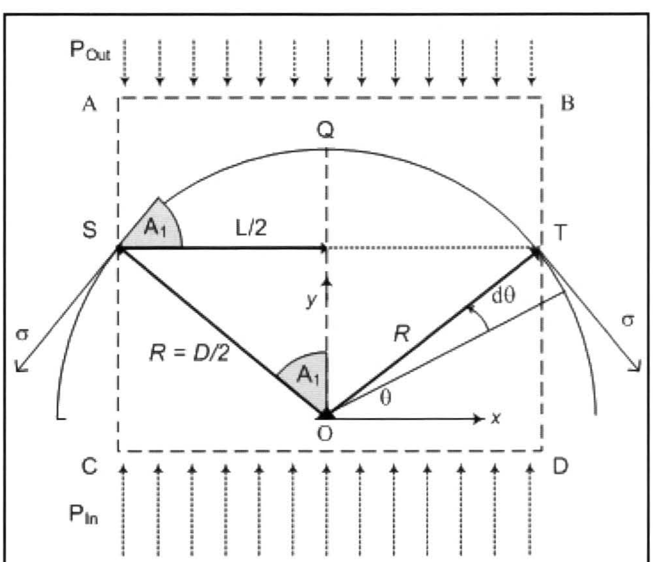


***Figure 2.** Forces acting on a cylindrical surface.*

where P_{In} is the pressure inside the circular boundary, P_{Out} is the pressure outside the circular boundary, σ is the surface tension, and R is the constant radius of the circle.

Eq. (1) does not apply to the curved surfaces that are produced when an applied force causes an object to bend so that the outside surface is stretched while the inside surface is compressed, for example when bending a piece of sheet metal. The sketch results presented later on assume further that $P_{Out} = 0$ for simplicity of presentation.

The laws of equilibrium can be applied to a thin-walled pressure vessel or to a small liquid drop. Figure 1 shows each of these systems, while Figure 2 shows the terminology for forces acting on a cylindrical surface. For a thin-walled pressure vessel, represented by arc SQT, the external inward pressure is distributed over the outside of the vessel while the internal outward pressure is distributed over the inside of the vessel. For a small cylindrical liquid drop, on the other hand, the body for the force balance is segment SQTS. The external inward pressure is distributed along arc SQT but the internal outward pressure is distributed along the line ST. Since gravitational forces are ignored, the force balance is the same in these two situations and the net force due to pressure can be found from a force balance on the control surface ABDC.

Consider a circle of radius R with the object on which the forces act defined by the position of a chord. The angle $2A_1$ subtended by the chord (Figure 2) determines the line of action of the pressure- and surface-tension-related forces and is given by

$$A_1 = \arcsin\left[\frac{L/2}{D/2}\right] = \arcsin\left[\frac{L}{D}\right] \tag{2}$$

where L is the length of the chord and D is the diameter of the circle.

Forces due to surface tension and net pressure are applied to the system and are quantified as follows:

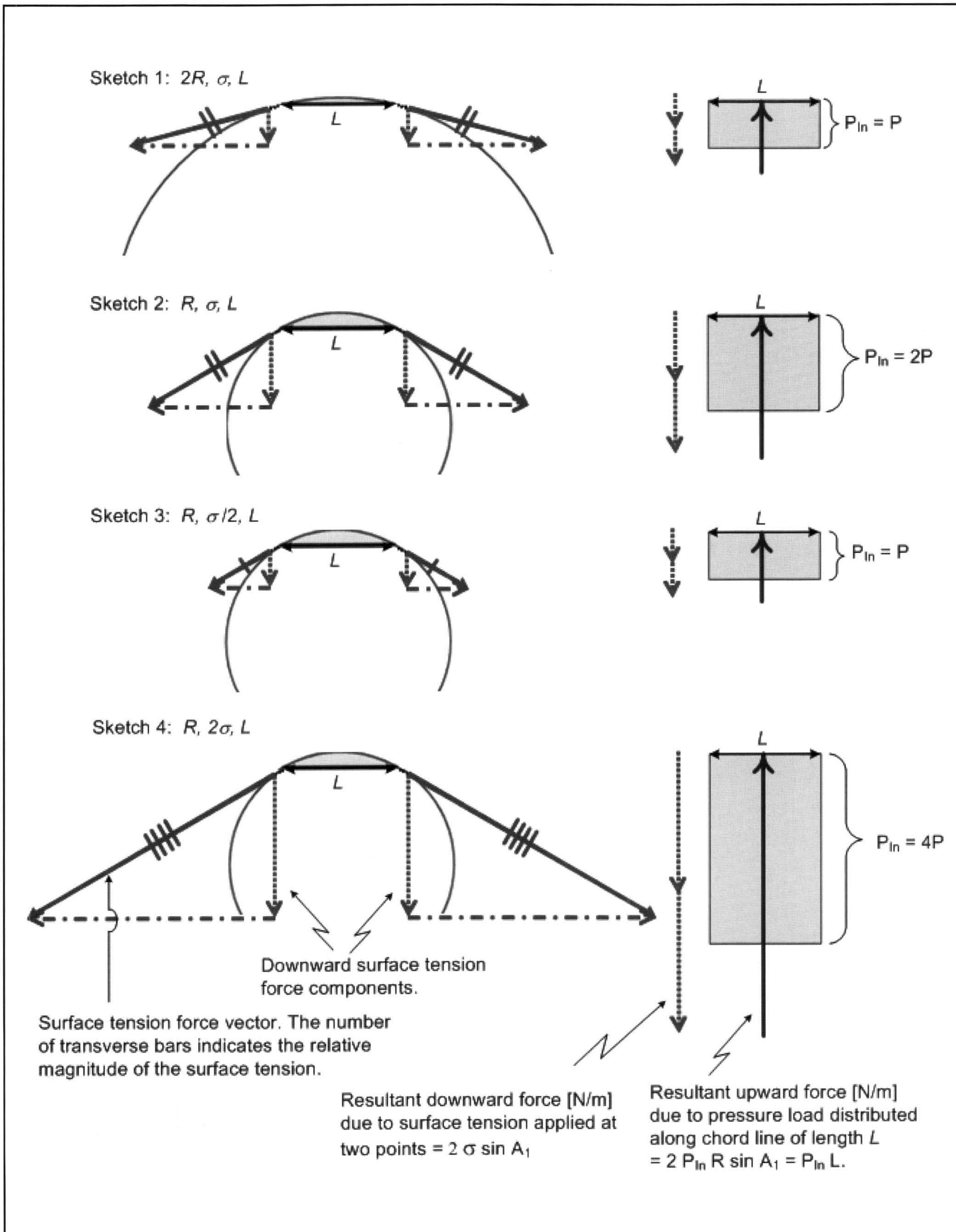
- **Surface Tension Force:** The force due to surface tension is applied as a pulling force tangent to curve SQT at points S and T in Figure 2. In an axisymmetric system, the horizontal x-components of the surface tension force cancel out while the vertical y-components both act downward and are added. The resultant downward force is

$$F_{Down} = 2\sigma \sin A_1 \tag{3}$$

- **Pressure Force:** A force balance on the control surface ABDC gives the resultant upward force

$$F_{Up} = 2R\left(P_{In} - P_{Out}\right)\sin A_1 \tag{4}$$

Eq. (4) is derived in the Appendix. Equating F_{Up} and F_{Down}


Sketch 1: 2R, σ, L
L
L
P_{In} = P
Sketch 2: R, σ, L
L
L
P_{In} = 2P
Sketch 3: R, σ/2, L
L
L
P_{In} = P
Sketch 4: R, 2σ, L
L
L
P_{In} = 4P
Downward surface tension force components.
Surface tension force vector. The number of transverse bars indicates the relative magnitude of the surface tension.
Resultant downward force [N/m] due to surface tension applied at two points = 2 σ sin A_1
Resultant upward force [N/m] due to pressure load distributed along chord line of length L = 2 P_{In} R sin A_1 = P_{In} L.

balances the acting forces and gives Eq. (1).

SKETCH RESULTS

This section presents examples of the scaled sketches generated from running the sketching program with different choices of variables. The sketches are discussed in terms of droplets but illustrate the situation for thin-walled pressure vessels as well. Sketches 1 to 7 in Figures 3 and 4 were generated in an Excel worksheet using a Visual Basic program.

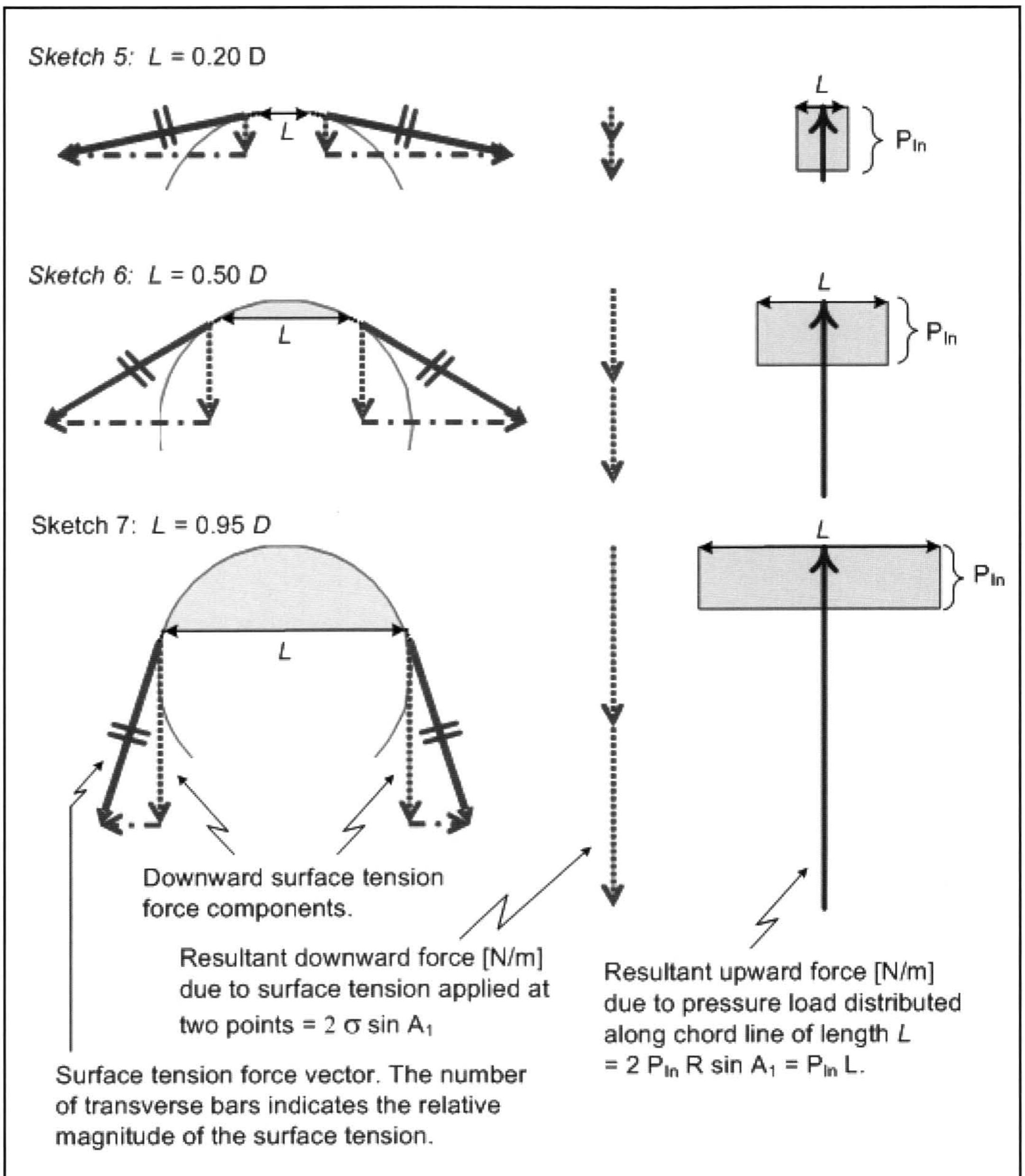
Each sketch shows the object (a segment of a circle) on which the forces act, with the points of application of the surface tension shown. To the right of the circle, the y-components of the surface tension force are added, following Eq. (3). The resultant upward force is shown by a vector equal in magnitude to F_{Down} but opposite in direction.

The pressure inside the object, P_{In}, and its application length L are shown superimposed on the resultant upward force F_{Up}. A convenient scale is chosen to illustrate the pressure inside the drop; this does not depend on the scaling of the force vectors since pressure is a scalar quantity.

Different Droplet Sizes, R and R/2

The oil inside a small droplet in a sample of mayonnaise (an oil-in-water emulsion) is under more pressure than the oil in a larger droplet, as can be seen by comparing Sketches 1 and 2 in Figure 3. The student can generate such sketches by running the spreadsheet program with values of R and 2R.

Walker[8] similarly illustrates that the y-components of the



Figures 3 and 4. *Scaled sketches automatically drawn in an Excel worksheet using a Visual Basic program. Students can produce such sketches along with the usual tabular values: in* ***Figure 3*** *(left) the sketches are used to visualize the forces acting on small and large drops and drops with the same radius but different surface tension values; in* ***Figure 4*** *(right) the sketches are used to visualize the forces acting when different portions of the same object are chosen.*

surface tension are shorter in a larger drop. The scaling of the sketches in Figure 3 is intended to help quantify the relationship: the internal pressure is doubled when the droplet radius is halved.

Different Surface Tension Values, σ/2, σ, 2σ

Comparing Sketches 3, 2, and 4 in Figure 3 illustrates the forces acting on a droplet of radius R and chord length L with the three different surface tension values σ/2, σ, and 2σ. If the surface tension is increased, the pressure inside the droplet is also increased.

Force Balance on Different Portions of the Same Object

The portion of the object on which the forces act is determined by choosing the different values of L; however, the calculated pressure, P_{In} must be the same. If the object is not a semicircle, the force vector representing the surface tension must be drawn with its components in the x and y directions and is not as easy to sketch accurately freehand. Figure 4 shows the forces acting on three different portions of the same object of radius R and with uniform surface tension σ. The y-component of the surface tension force increases when longer chord lengths L are chosen, so that the forces balance and the calculated pressure inside the drop is the same.

APPLICATIONS

Small Liquid Drops and Soap Bubbles

Soap bubbles have two air-liquid interfaces, one inside and one outside the liquid film, as illustrated by Guyon, *et al.*[4] The tension to be used in Eq. (1) is therefore twice the surface tension between the air and the liquid:

$$\sigma_1\left[\frac{N}{m}\right]=2\sigma\left[\frac{N}{m}\right] \tag{5}$$

The difference between the situation for a small liquid drop and a small air bubble can be visualized by comparing Sketches 2 and 4 in Figure 3.

Balloons and Alveoli

The inflation and deflation of balloons, and of the terminal air sacs of the lungs, are processes in which radius and wall tension change continuously, and hence the instantaneous pressure-difference varies. To inflate a balloon, a volume of air must flow into the balloon. As the balloon expands, its volume increases and the membrane stretches. The first puff when inflating a balloon must apply the highest pressure since the radius of the film is smallest, as shown by Eq. (1). The wall tension itself varies during inflation, increasing as the membrane becomes more stretched.

Two interconnected alveoli of different size form an unstable system. If the pressure were higher in the smaller alveolus, air would flow out into the larger alveolus, eventually leading to the collapse of the smaller alveolus. This situation would arise if the surface tension were the same in each alveolus, illustrated by Sketches 1 and 2 in Figure 3. Fortunately, a difference in surface tension due to the presence of lung surfactant allows small alveoli to be connected to large alveoli without collapsing (Sketches 1 and 3 in Figure 3). The magnitudes of the pressures and tensions involved could be explored using alveolus radii in the range 0.05 to 0.10 mm[5] and surface tensions in the range of 1 $mN \cdot m^{-1}$ to 28 $mN \cdot m^{-1}$.[5]

Thin-Walled Pressure Vessels

Eq. (1) can be used to find the wall tension in thin-walled pressure vessels when the wall is in pure tension and no shear stresses are present in the wall. If an allowable tensile stress is specified by the choice of construction material, the allowable wall tension, σ_2, is

$$\sigma_2\left[\frac{N}{m}\right]=S\left[\frac{N}{m^2}\right]T[m] \tag{6}$$

in which S is the tensile stress and T is the wall thickness. The maximum pressure that can be sustained inside a pressure vessel of a given radius, that has a given wall thickness and a known allowable tensile stress, can be calculated by combining Eqs. (1) and (6) and solving for ($P_{In} - P_{Out}$).

Notice in Eq. (1) that the pressure difference and the surface tension are linearly related

$$\sigma_2\left[\frac{N}{m}\right]=\Delta P\left[\frac{N}{m^2}\right]R[m] \tag{7}$$

where R is the radius of a tank. Eq. (7) shows that when using a large tank, the wall tension required to withstand a given internal design pressure is also large.[6] Sketches 1 (small tank) and 3 (large tank) in Figure 3 with the same internal pressure illustrate this situation. Thus, several smaller tanks might be

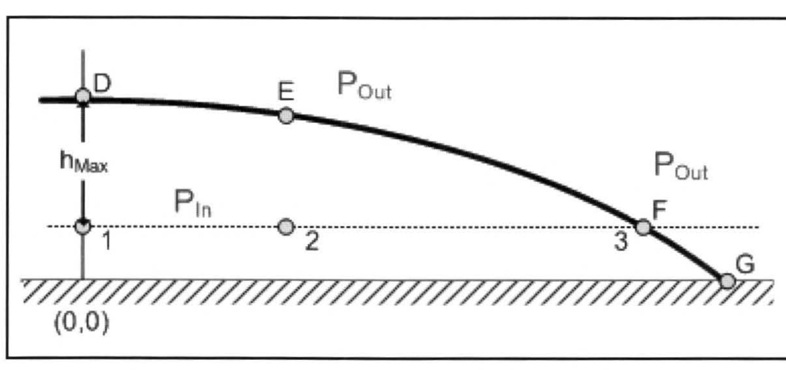


Figure 5. *Pressure inside a noncircular drop at rest on a solid surface.*

more suited to store a given volume while keeping within the limits of the allowable tensile stress for the material. Liquefied gas is often transported in batteries of smaller pressure vessels, allowing the pressure to be larger within each cylinder for the same material of construction.

Compression Bandages

While not a usual study topic in chemical engineering, the compression of a limb beneath a bandage is related to the tension in the bandage. The more tightly the bandage is wrapped, the greater the compression applied. To calculate the compression, given by P_{In} in Eq. (1), we need to know the tension in the bandage, which is given by[7]

$$\sigma_3\left[\frac{N}{m}\right]=\frac{j[-]F_B[N]}{W[m]} \qquad (8)$$

in which j layers of a bandage of width W are applied to a limb with a force of F_B newtons. The radius in Eq. (1) is calculated from the circumference of the patient's leg. Because this radius is not constant, the compression will vary in practice.[7] Sketches 1 and 2 in Figure 3 illustrate that a child's leg of radius R would experience a much higher compression than an adult's when applying a bandage with the same tension.

The calculated compression can be compared to a standard diastolic blood pressure of 80 mm Hg (10.7 kilopascals) to ensure that blood can still flow freely.

Large Liquid Drops

A liquid drop adjusts its shape in response to differences in pressure and will not remain spherical under gravity. Figure 5 is a sketch of a large liquid drop with a noncircular cross-section resting on a solid surface.

Assuming a uniform pressure outside the drop, the pressure at point "1" in Figure 5, located at depth h_{Max} below the apex of the liquid drop of density ρ_{in}, is given by

$$P_{In}=P_{Out}+\frac{\sigma}{b}+\rho_{In}\,gh_{Max} \qquad (9)$$

Eq. (9) uses the radius of curvature at the apex of the drop, b, since the radius of curvature is no longer the same at each point on the surface but changes continuously along the drop boundary. For a static liquid, the pressure is constant as we traverse the path 1-2-3 at constant elevation. Since the height of liquid above points 1-2-3 decreases, the radius of curvature must decrease as we proceed from D to E to F. The pressure is not the same everywhere inside the object shown in Figure 5, but is higher at greater depths below the air-liquid boundary.

Note in Figure 5 that three phases are in contact at point G, where the air-liquid boundary of the drop meets the solid surface; the tangent angle measured through the liquid phase is known as the contact angle. Introducing the concepts of surface tension and contact angle separately might help a student make the distinction between the separate phenomena of the presence of tension at a boundary and wetting phenomena where a liquid is in contact with a solid surface.[9] Electrowetting, in which an electric field modifies the contact angle of a liquid droplet in contact with an insulated conductor, is useful in making lenses with no mechanical moving parts in digital camera technology.[1]

SUMMARY

This paper has presented a suggestion of visualizing the action of surface tension by using a scaled sketch. By producing a scaled sketch along with calculated values, the approach would be a useful supplement for visual learners. The additional applications where forces act over curved surfaces are intended to provide an idea of how the concept of tension at a boundary is widely useful. A scaled sketch would also be a useful supplement to spreadsheet programs for drop size and shape distributions as well as for other simple force balances.

ACKNOWLEDGMENTS

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NOMENCLATURE

- A_1 Half the angle subtended by a chord of length L, [radians]
- b Radius of curvature at the apex of an axisymmetric drop, [m]
- D Diameter of a circle, [m]
- F_{Down} Force per unit length downward, [$N \cdot m^{-1}$]
- F_{Up} Force per unit length upward, [$N \cdot m^{-1}$]
- F_{Out} External force per unit length acting inward, [$N \cdot m^{-1}$]
- F_{In} Internal force per unit length acting outward, [$N \cdot m^{-1}$]
- F_B Force with which a bandage is applied, [N]
- g Acceleration due to gravity, [$m \cdot s^{-2}$]
- h_{Max} Height of liquid inside a drop at the apex of the drop, [m]
- j Number of times a bandage is wrapped around a limb, [-]
- L Chord length, [m].
- ΔP Pressure difference across a boundary equal to $P_{In} - P_{Out}$, [$N \cdot m^{-2}$]
- P_{In} Pressure inside the circle, on the concave side, [$N \cdot m^{-2}$]
- P_{Out} Pressure outside the circle, on the convex side, [$N \cdot m^{-2}$]
- R Radius of a circle, radius of curvature, [m]
- s Length of segment of curve, [m]
- S Tensile stress, [$N \cdot m^{-2}$]
- T Wall thickness, [m]
- W Width of a tensor bandage, [m]
- x Coordinate in the horizontal direction, [m]
- y Coordinate in the vertical direction, [m]
- θ Coordinate in the angular direction, [radians]
- ρ_{In} Density inside the boundary of a liquid droplet, [$kg \cdot m^{-3}$]
- σ Surface tension, [$N \cdot m^{-1}$]
- σ_i Surface tension for different situations, [$N \cdot m^{-1}$]

APPENDIX — *DERIVATION OF EQ. (4)*

The external inward force on a differential element of length ds of arc SQT in Figure 2 is

$$dF_{Out} = P_{Out} ds \qquad (10)$$

The y-component of the force in Eq. (10) is

$$dF_{y,Out} = -P_{Out} \sin\theta \, ds = -P_{Out} \sin\theta \, d(R\theta) = -P_{Out} R \sin\theta \, d\theta \qquad (11)$$

$$F_{y,Out} = \int dF_{y,Out} = -P_{Out} R \int_{(\pi/2)-A_1}^{(\pi/2)+A_1} \sin\theta d\theta = P_{Out} R[\cos\theta]_{(\pi/2)-A_1}^{(\pi/2)+A_1} \qquad (12)$$

Evaluating the integral in Eq. (12) and substituting $L = 2R\sin A_1$ gives the y-component of the external inward force on SQT

$$F_{y,Out,SQT} = P_{Out} R\{-\sin A_1 - \sin A_1\} = -2P_{Out} R \sin A_1 = -P_{Out} L \qquad (13)$$

The internal outward force on a differential element of length ds of arc SQT in Figure 2 is

$$dF_{In} = P_{In} ds \qquad (14)$$

The y-component of the force in Eq. (14) is found using the same procedure as in Eqs. (11) and (12)

$$F_{y,In,SQT} = -P_{In} R[\cos\theta]_{(\pi/2)-A_1}^{(\pi/2)+A_1} = 2P_{In} R \sin A_1 = P_{In} L \qquad (15)$$

For a liquid droplet, the internal outwards force along the line ST can be immediately written by examination of Figure 2:

$$F_{y,In,ST} = P_{In} L = 2P_{In} R \sin A_1 \qquad (16)$$

Thus the net force due to pressure is the same whether the object is a thin-walled pressure vessel or a small liquid drop (the two cases illustrated in Figure 1). Adding the pair of equations (15) + (13) or (16) + (13) gives Equation (4):

$$F_{Up} = 2R(P_{In} - P_{Out}) \sin A_1 \qquad (4)$$

Many students find visual information such as pictures and diagrams useful,[3] and sketches are commonly used when presenting the concept of surface tension. While unscaled sketches are helpful, scaled sketches can more accurately illustrate quantitative relationships.

REFERENCES

1. "Going With the Flow," *The Economist Technology Quarterly,* 27-28 (June 12, 2004)
2. Adamson, A.W., *Physical Chemistry of Surfaces*, 4th Ed., John Wiley and Sons, New York (1982)
3. Felder, R.M., and L. Silverman, "Learning and Teaching Styles in Engineering Education," *Eng. Ed.*, **78**, 674 (1988)
4. Guyon, E., J. Prost, C. Betrencourt, C. Boulet, and B. Volochine, "Attention aux Tensions Superficielles!" *European J. of Physics,* **3**, 159 (1982)
5. Liu, M., "Synchronized Changing of Transinterface Pressure, Bubble Radius, and Surface Tension: A Unique Feature of Lung Surfactant," *Chemistry and Physics of Lipids* **89**, 55 (1997)
6. Nave, R., *Alveoli of the Lungs*, <http://hyperphysics.phy-astr.gsu.edu /hbase/hframe.html>, Department of Physics and Astronomy, Georgia State University, Mar. 2004. Last accessed July 11, 2005.
7. Thomas, S., "The Use of the Laplace Equation in the Calculation of Sub-Bandage Pressure," *European Wound Management Association J.*, **3**, 21 (2003)
8. Walker, J., "Why are the First Few Puffs the Hardest When You Blow Up a Balloon?" *Scientific American,* 136, Dec. (1989)
9. Walton, A., "Surface Tension and Capillary Rise," *Physics Education,* **7**, 491 (1972) ❐