FORCED CONVECTION HEAT TRANSFER IN CIRCULAR PIPES

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Forced convection inside circular pipes under fully developed conditions is one of the main subjects covered in both undergraduate- and graduate-level heat transfer courses. Two types of boundary conditions are usually considered, *i.e.*, constant wall heat flux and constant wall temperature. In engineering calculations, heat transfer correlations are expressed in terms of the Nusselt number and such expressions require the solution of the energy equation given as

$$\rho \hat{C}_P v_z \frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad (1)$$

in which the velocity distribution under laminar flow conditions is given by

$$v_z = 2\langle v_z \rangle \left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (2)$$

When the heat flux at the wall is constant, the temperature gradient in the axial direction, $\partial T/\partial z$, is constant. This makes the solution of Eq. (1) rather simple since the left side is dependent only on r. Integration of Eq. (1) twice yields the Nusselt number equal to 48/11. This approach is presented in almost all textbooks on heat transfer and/or transport phenomena.

In the case of constant wall temperature, however, the solution of Eq. (1) requires advanced mathematical skills in partial differential equations.[1] As a result of this mathematical complexity, the value of the Nusselt number is given as 3.66 in textbooks without presenting the analysis. Incropera and

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DeWitt,[2] for example, stated that:

". . . the solution may be obtained by an iterative procedure, which involves making successive approximations to the temperature profile. The resulting profile is not described by a simple algebraic expression, but the resulting Nusselt number may be shown as Nu = 3.66."

The method of Stodola and Vianello[3, 4] is an approximate technique used for a quick estimation of the first eigenvalue. The purpose of this paper is to show students how to apply this technique in the calculation of the Nusselt number for forced convection in a circular pipe when the wall temperature is constant. From my experience in teaching graduate-level Transport Phenomena and Heat Transfer courses, the method is well received by students.

MATHEMATICAL ANALYSIS

Consider the laminar flow of an incompressible Newtonian fluid in a circular pipe of radius R under the action of a pressure gradient. The fluid is at a uniform temperature of T_o for $z < 0$. For $z > 0$, the wall temperature is kept constant at T_w ($T_w > T_o$) and we want to develop a correlation for heat transfer in terms of the Nusselt number under thermally fully developed conditions.

BULK TEMPERATURE GOVERNING EQUATION

As engineers, we are interested in the variation of the bulk (or, mean) fluid temperature, T_b, rather than the local temperature, T. The bulk fluid temperature is defined by

$$T_b = \frac{\int_0^{2\pi}\int_0^R v_z T\, r\, dr d\theta}{\int_0^{2\pi}\int_0^R v_z\, r\, dr d\theta} = \frac{1}{\pi R^2 \langle v_z \rangle}\int_0^{2\pi}\int_0^R v_z T\, r\, dr d\theta \tag{3}$$

Since both v_z and T are independent of θ, Eq. (3) simplifies to

$$T_b = \frac{2}{R^2 \langle v_z \rangle}\int_0^R v_z T\, r\, dr \tag{4}$$

The governing equation for the bulk temperature can be obtained by multiplying Eq. (1) by r dr and integrating from r = 0 to r = R, *i.e.*,

$$\rho \hat{C}_P \int_0^R v_z \frac{\partial T}{\partial z} r\, dr = k\int_0^R \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) r\, dr \tag{5}$$

Since $v_z \neq v_z(z)$, the integral on the left side of Eq. (5) can be rearranged, with the help of Eq. (3), as

$$\rho \hat{C}_P \int_0^R v_z \frac{\partial T}{\partial z} r\, dr = \rho \hat{C}_P \int_0^R \frac{\partial (v_z T)}{\partial z} r\, dr = \rho \hat{C}_P \frac{d}{dz}\left(\int_0^R v_z T\, r\, dr\right) = \rho \hat{C}_P \left(\frac{R^2 \langle v_z \rangle}{2}\right)\frac{dT_b}{dz} \tag{6}$$

On the other hand, the integral on the right side of Eq. (5) takes the form

$$k\int_0^R \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) r\, dr = k\left(r\frac{\partial T}{\partial r}\right)\Bigg|_{r=0}^{r=R} \tag{7}$$

The heat flux at the wall, q_w, is defined by

$$q_w = k\frac{\partial T}{\partial r}\Bigg|_{r=R} \tag{8}$$

so that Eq. (7) becomes

$$k\int_0^R \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) r\, dr = R\, q_w \tag{9}$$

Substitution of Eqs. (6) and (9) into Eq. (5) results in the governing equation for the bulk temperature as

$$\frac{dT_b}{dz} = \frac{2\, q_w}{\rho \hat{C}_P R \langle v_z \rangle} \tag{10}$$

THERMALLY FULLY DEVELOPED CONDITIONS

The requirement for a thermally fully developed flow is expressed as

$$\frac{\partial}{\partial z}\left(\frac{T - T_b}{T_w - T_b}\right) = 0 \tag{11}$$

Note that the thermally fully developed condition also implies that the local heat transfer coefficient, h, is a constant.

When the wall temperature, T_w, is constant, Eq. (11) reduces to

$$\frac{\partial T}{\partial z} = \left(\frac{T_w - T}{T_w - T_b}\right)\frac{dT_b}{dz} \tag{12}$$

Substitution of Eq. (10) into Eq. (12) results in

$$\frac{\partial T}{\partial z} = \left(\frac{T_w - T}{T_w - T_b}\right)\frac{2\, q_w}{\rho \hat{C}_P R \langle v_z \rangle} \tag{13}$$

NUSSELT NUMBER FOR CONSTANT WALL TEMPERATURE

The Nusselt number is defined by

$$Nu = \frac{hD}{k} \tag{14}$$

The heat flux at the wall is also expressed in terms of the Newton's law of cooling as

$$q_w = h(T_w - T_b) \tag{15}$$

so that the Nusselt number takes the form

$$Nu = \frac{[q_w / (T_w - T_b)]2R}{k} \tag{16}$$

Elimination of the term q_w between Eqs. (13) and (16) leads to

$$\frac{\partial T}{\partial z} = \frac{Nu(T_w - T)k}{\rho \hat{C}_P R^2 \langle v_z \rangle} \tag{17}$$

Substitution of Eqs. (2) and (17) into Eq. (1) yields

$$2Nu\left[1-\left(\frac{r}{R}\right)^2\right](T_w - T) = \frac{R^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{18}$$

The boundary conditions associated with Eq. (18) are

$$\begin{aligned} &\text{at } z = 0 && T = T_o \\ &\text{at } r = 0 && \frac{\partial T}{\partial r} = 0 \\ &\text{at } r = R && T = T_w \end{aligned} \tag{19}$$

In terms of the following dimensionless quantities

$$\theta = 1 - \frac{T - T_b}{T_w - T_b} = \frac{T_w - T}{T_w - T_b} \tag{20}$$

$$\xi = \frac{r}{R} \tag{21}$$

the governing equation together with the boundary conditions take the form

$$-2Nu\left(1-\xi^2\right)\theta = \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right) \tag{22}$$

$$\text{at } \xi = 0 \qquad \frac{d\theta}{d\xi} = 0 \tag{23}$$

$$\text{at } \xi = 1 \qquad \theta = 0 \tag{24}$$

It should be kept in mind that the dimensionless temperature, θ, is dependent on only the dimensionless radial coordinate, ξ, for the thermally fully developed condition. Eq. (22) can be easily solved for Nu by the method of Stodola and Vianello.

THE METHOD OF STODOLA AND VIANELLO

The method of Stodola and Vianello[3, 4] is a successive approximation technique to estimate the lowest eigenvalue λ_1 in the boundary value problem of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] = -\lambda w(x) y \tag{25}$$

with appropriate homogeneous boundary conditions at x = a and x = b. The procedure is as follows:

1. Assume a trial function for $y_1(x)$ which satisfies the boundary conditions x = a and x = b.
2. On the right side of Eq. (25), replace y(x) by $y_1(x)$.
3. Solve the resulting differential equation and express the solution in the form

$$y(x) = \lambda f_1(x) \tag{26}$$

The first approximation to λ_1, $\lambda_1^{(1)}$, is given by

$$\lambda_1^{(1)} = \frac{\int_a^b w(x) f_1(x) y_1(x)\,dx}{\int_a^b w(x)[f_1(x)]^2 dx} \tag{27}$$

4. Repeat step (2) with a second trial function $y_2(x)$ defined by

$$y_2(x) = f_1(x) \tag{28}$$

5. Solve the resulting differential equation and express the solution in the form

$$y(x) = \lambda f_2(x) \tag{29}$$

The second approximation to λ_1, $\lambda_1^{(2)}$, is given by

$$\lambda_1^{(2)} = \frac{\int_a^b w(x) f_2(x) y_2(x)\,dx}{\int_a^b w(x)[f_2(x)]^2\,dx} \tag{30}$$

6. Continue the process until the desired convergence is obtained.

For the problem at hand, comparison of Eq. (22) with Eq. (25) gives

$$y = \theta \qquad x = \xi \qquad p = \xi \qquad \lambda = Nu \quad w = 2\xi(1-\xi^2) \tag{31}$$

A reasonable first guess for θ which satisfies the boundary conditions is

$$\theta_1(\xi) = 1 - \xi^2 \tag{32}$$

Substitution of Eq. (32) into the left-side of Eq. (22) gives

$$\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right) = -2Nu\left(\xi - 2\xi^3 + \xi^5\right) \tag{33}$$

The solution of Eq. (33) using the boundary conditions given by Eqs. (23) and (24) is

$$\theta = Nu\underbrace{\left(\frac{11 - 18\xi^2 + 9\xi^4 - 2\xi^6}{36}\right)}_{f_1(\xi)} \tag{34}$$

The first approximation to the Nusselt number is calculated from Eq. (27) as

$$Nu^{(1)} = \frac{\int_0^1 \xi(1-\xi^2)^2 f_1(\xi)d\xi}{\int_0^1 \xi(1-\xi^2) f_1^2(\xi)d\xi} \tag{35}$$

Substitution of $f_1(\xi)$ from Eq. (34) into Eq. (35) and evaluation of the integrals gives

$$Nu^{(1)} = 3.663 \tag{36}$$

The trial function for the second approximation is

$$\theta_2(\xi) = \frac{11 - 18\xi^2 + 9\xi^4 - 2\xi^6}{36} \tag{37}$$

Substitution of Eq. (37) into the left-side of Eq. (22) gives

$$\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right)=-\frac{Nu}{18}(11\xi-29\xi^3+27\xi^5-11\xi^7+2\xi^9) \quad (38)$$

The solution of Eq. (38) using the boundary conditions given by Eqs. (23) and (24) is

$$\theta=Nu\underbrace{\left(\frac{2457-4400\xi^2+2900\xi^4-1200\xi^6+275\xi^8-32\xi^{10}}{28,800}\right)}_{f_2(\xi)} \quad (39)$$

Therefore, the second approximation to the Nusselt number is given by

$$Nu^{(2)}=\frac{\int_0^1\xi(1-\xi^2)f_2(\xi)\theta_2(\xi)\,d\xi}{\int_0^1\xi(1-\xi^2)f_2^2(\xi)\,d\xi} \quad (40)$$

Substitution of $f_2(\xi)$ and $\theta_2(\xi)$ from Eqs. (39) and (37), respectively, into Eq. (40) and evaluation of the integrals gives

$$Nu^{(2)}=3.657 \quad (41)$$

which is equal to the exact value calculated by Graetz and Nusselt.

The solution of eigenvalue problems by the method of Stodola and Vianello gives accurate results and convergence is very rapid. Although the integrals seem formidable, they can be easily evaluated using engineering calculation software such as MATHEMATICA or MATHCAD. The method is easy to follow and students have no difficulty in understanding the presented material. The transformation of the governing differential equation, Eq. (1), to the form to which the method of Stodola and Vianello is applied, Eq. (18) or Eq. (22), is also very helpful for students in grasping the concept of area averaging and the difference between local and bulk temperatures, as well as their functional dependence on coordinate directions.

NOMENCLATURE

$\hat{C}_P$	Heat capacity, J/ kg·K
D	Pipe diameter, m
k	Thermal conductivity, W/m·K
Nu	Nusselt number, dimensionless
q_w	Heat flux at the wall, W/m^2
R	Pipe radius, m
r	Radial coordinate, m
T	Temperature, K
T_b	Bulk temperature, K
T_w	Wall temperature, K
v_z	Axial velocity, m/s
$\langle v_z \rangle$	Axial average velocity, m/s
z	Axial coordinate, m

Greek symbols

θ	Dimensionless temperature
ξ	Dimensionless radial coordinate
ϱ	Density, kg/ m^3

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