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A Holographic Quantum Code

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Abstract

It was shown by [2] how bulk operators in the AdS/CFT correspondence can be represented on the boundary analogously to the way logical qubits are represented in an encoded subspace in quantum error correction. Then in [1] holographic tensor networks that serve as toy models of the bulk boundary correspondence were introduced. This paper reviews some of the developments of [1] and [2]. Then, it is demonstrated explicitly how to construct perfect tensors, which are essential to the tensor networks mentioned in [2]. Lastly, a new example of a holographic quantum error-correcting code based on an eight index perfect tensor is presented.

Introduction

In the Anti-de Sitter Conformal Field Theory Correspondence, a standard result is that a bulk operator $\phi(x)$ can be represented as $\phi(x) = \int dYK(x;Y)O(Y)$ to leading order in $\frac{1}{N}$ where x is a bulk point and the integral is taken over the boundary. The smearing function $K(x;Y)$ can be chosen to have support only when x and Y are spacelike separated. Thus only a portion of the boundary is needed to construct a specific operator. This is depicted in Figure 1 for AdS_3, where any local bulk operator in the blue area can be represented on the intersection of the boundary of the blue area with the boundary of the entire space.

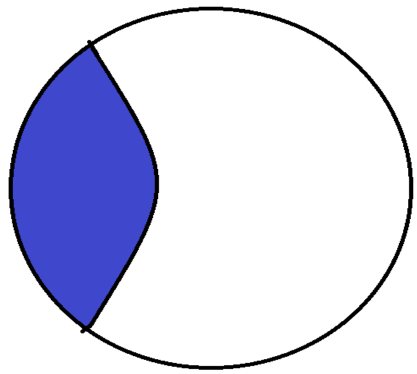

Figure 1. Rindler Wedge in AdS Space. Any bulk local bulk operator in the blue area can be constructed to have support on intersection of the boundary of the blue area with the boundary of the entire space. The interior lines are lightlike geodesics.

The requirement of only a portion of a boundary to reconstruct an operator is reminiscent of quantum error correction as pointed out in [2]. A quantum error correcting code typically maps a state from one space to a state in a larger space such that there is some redundancy in the encoded state. This allows for an operator acting only on a portion of a state to have the same action as an operator acting on the entire state, which is analogous to the reconstruction of bulk operators on the boundary explained above. The holographic codes proposed in [1] demonstrate this analogy. The encoding maps, when viewed as tensor networks, closely resemble the picture of AdS_3 pictured above. The tensors used in these networks are perfect tensors, which will be explained below. Then, a link between perfect tensors and classical coding theory is demonstrated. Lastly, a particular perfect tensor is be presented and a holographic code constructed from it.

AME States and Perfect Tensors

To understand the tensor networks proposed in [1], we must first explain perfect tensors. An isometry $T: H \rightarrow H'$ is a linear mapping from one Hilbert space to another that preserves the inner product. This is only possible if the dimension of H is less than or equal to the dimension of H'. T can be viewed as a two index tensor, which we call an isometric tensor. The H index is the incoming index and the H' index is the outgoing index. isometric tensors have the property that an operator acting on its incoming index can be replaced by an equal norm operator acting on its outgoing index. That is $TO = O'T$ for some O'.

An important class of isometric tensors is perfect tensors. A $2n$ index tensor is perfect if any bi-partition of its indices results in an isometric tensor from the set of smaller indices to the set of larger indices. This property allows one to push an operator acting on an incoming index through a $2k$ index perfect tensor onto k indices. This is demonstrated in Figure 2.

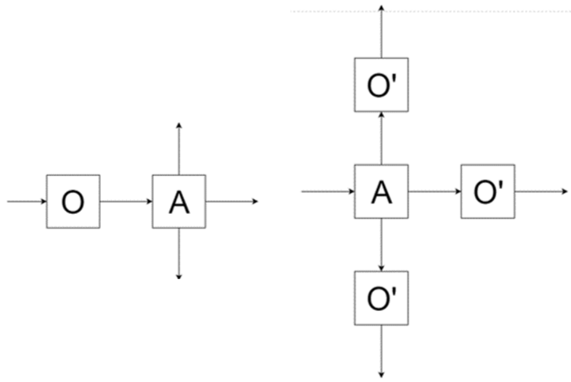

Figure 2. A Tensor network depiction of a perfect tensor with an operator acting on its incoming index. Each square represents a tensor and each arrow/line an index. Here A is a perfect tensor, so the diagrams are equivalent. The left diagram shows an operator acting on the incoming index of A and the right diagram shows an operator acting on the outgoing indices of A. If A were perfect the right diagram could actually be chosen to have an operator acting on only two of the indices.

There is a correspondence between perfect tensors and quantum information. An Absolutely Maximally Entangled (AME) state is a state $\sum_{j_i} c_{j_1 j_2 \dots j_n} |j_1, j_2, .. j_n\rangle$ in $H = H_1 \otimes H_2 \otimes \dots \otimes H_n$with the property that the reduced density matrix on $n' \leq \left\lfloor \frac{n}{2} \right\rfloor$ parties is maximally mixed. That is the reduced density matrix is proportional to the identity. If n is even, then the tensor $c_{j_1, j_2, \dots j_n}$ is a perfect tensor. We label an AME state as $AME(n, q)$ where n is the number of parties and $q$is the dimension of the H_i, called the local dimension. A qubit is just an element of a two dimensional Hilbert Space and a qudit is an element of a d dimensional space. Therefore an $AME(n, d)$ state has n qudits.

Perhaps the most well known example of an AME state is the EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Another example would be the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

A particular class of AME states is minimal support AME states. Minimal support AME states have support on $q^{h\left(\frac{n}{2}\right)}$ states, where $h(x)$ is the greatest integer less than or equal to x. There is a correspondence between minimal support AME states and maximum distance separating (MDS) codes, which will now be introduced.

Coding Theory and MDS Codes

A linear code is a subspace of order q^k of $GF(q)^n$ where $GF(q)$ is the Galois field of order q and thus $GF(q)^n$ is the Cartesian product with itself n times. The elements in the subspace are called the words of the code. The distance between two words $x, y \in GF(q)^n$ is the number of non-zero elements in $x - y$. The code can be thought of as mapping elements from $GF(q)^k$ to the subspace in $GF(q)^n$. The distance of a code is the minimal distance between any two words in the code. The weight of a word is the number on nonzero components of the word. A code of subspace order q^k in $GF(q)^n$ with distance d is denoted $[n, k, d]$.

For a linear $[n, k, d]$ code, the generator matrix G is the matrix formed by taking the words of the code as rows. Thus G is a $k \times n$matrix and every code word can be written as vG, where $v \in GF(q)^k$.

The Singleton bound is a bound on classical codes that states for linear code $d \leq n - k + 1$, and a code saturating the bound is called a maximum distance separating (MDS) code. Examples of MDS codes include Reed-Solomon Codes and Generalized Reed-Solomon Codes. Details on these codes can be found in [4] and [5] respectively.

Given any $[n, k, d]$ MDS code over $GF(q)$ with generator matrix G, the state $\sum_{v \in GF(q)^k} |vG\rangle$ is a minimal support AME state. For finite fields of order p^k where $k \geq 2$, to translate the words into spin states, we interpret each letter to be a state in the tensor product $|p\rangle^k$, for example, the states $|0\rangle$,$|1\rangle$and $|x+1\rangle$ where $0, x+1 \in GF(4)$can be interpreted as $|00\rangle$,$|10\rangle$and $|11\rangle$respectively.

As proved in theorem 2 in [3] there always exists an n party AME state for a large enough local dimension. This can be proven through the use of Reed-Solomon codes. Therefore there exists a $2k$ index perfect tensor for any positive integer k.

Results

An eight party AME state can be constructed from $n = 8$ MDS code. One such code is the generalized Reed-Solomon Code $GRS_{8,4}$ over $GF(13)$ that has generator matrix

$$G = \begin{bmatrix} 1 & 3 & 4 & 5 & 8 & 9 & 10 & 12 \\ 1 & 9 & 3 & 12 & 12 & 3 & 9 & 1 \\ 1 & 1 & 12 & 8 & 5 & 1 & 12 & 12 \\ 1 & 3 & 9 & 1 & 1 & 9 & 3 & 1 \end{bmatrix}$$

Let C be the code word subspace of $\mathrm{GRS}_{8,4}$. Then the tensor A_v , where $A_v = 1$ if $v \in C$and $A_v = 0$ if $v \in GF(13)^n \setminus C$, is an eight index perfect tensor.

Thus we have a perfect tensor A with eight indices constructed from an MDS code. We can then construct a network using the (7 3 2) regular heptagon tiling of the hyperbolic plane shown in figure 3. We interpret each heptagon as an eight index perfect tensor with one uncontracted index going into the page the seven contracted indices as the blue figures. Since the network cannot go to infinity, the network must be truncated at some layer. If a boundary heptagon has one contracted index to the previous layer then it has six uncontracted indices. If a boundary heptagon has two contracted index to the previous layer then it has five uncontracted indices. The network is an isometry from the bulk indices to the uncontracted indices on the boundary.

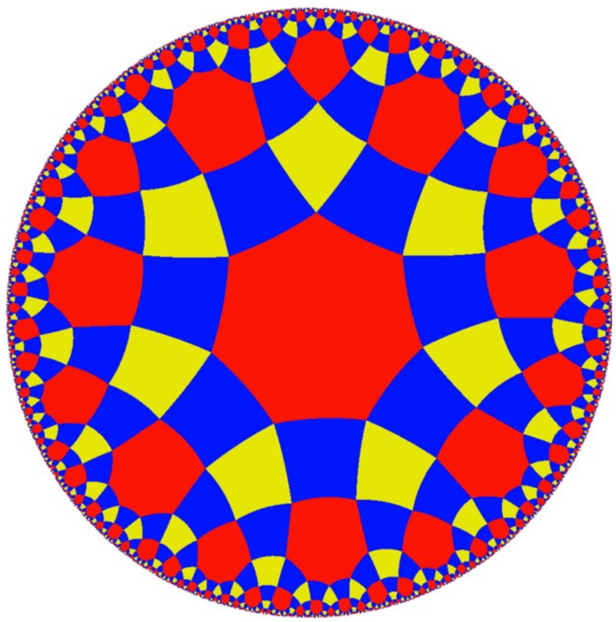

Figure 3. The (7 3 2) tiling of the hyperbolic plane. As a tensor network, each red heptagon represents a perfect tensor. Each heptagon has an uncontracted index going into the page that represents a bulk qudit. The blue areas represent a contracted index between two perfect tensors. At some point the number network is truncated, and the boundary heptagons then have uncontracted indices pointed outwards. These uncontracted indices are the physical qudits.

In the tiling the tensors are organized into layers. Call the center the zeroth layer. Let f_n be number of tensors at layer n with one index connection to the previous layer and let g_n be the number of tensors at layer n with two indices connected to the previous layer. Then f_n and g_n obey the recursion relation

$$\begin{bmatrix} f_{n+1} \\ g_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ g_n \end{bmatrix}$$

Since $f_1 = 7$ and $g_1 = 0$ the recursion is solved by

$$\begin{bmatrix} f_n \\ g_n \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

If the code has a total of n layers then it has $N_{boundary} = 6f_n + 5g_n$ qudits on the boundary. The number of bulk qudits is

$$N_{bulk} = 1 + \sum_{i=1}^{n} (f_i + g_i)$$

For large n

$$\begin{bmatrix} f_n \\ g_n \end{bmatrix} = \begin{bmatrix} \frac{7+\sqrt{21}}{2}(\frac{5+\sqrt{21}}{2})^n \\ \sqrt{\frac{7}{3}}(\frac{5+\sqrt{21}}{2})^n \end{bmatrix} + \mathcal{O}((\frac{5-\sqrt{21}}{2})^n)$$

It is then possible to calculate the asymptotic rate to be approximately

$$\lim_{n\to\infty} \frac{N_{bulk}}{N_{boundary}} = \frac{1}{\sqrt{21}}$$

Given an operator acting on the center qudit what is the minimum portion of the boundary needed to construct the operator on the boundary? Each heptagon has five or six outgoing legs and only four outgoing indices are required to push the operator onto, hence there is a choice of which legs to push the operator on. We can always make the best choices and thus recover a narrower wedge that the operator is pushed through.

If we truncate at the nth layer then the number of boundary heptagons with boundary indices which the operator will have support on satisfies the following recursion relation

$$\begin{bmatrix} f_{n+1} \\ g_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ g_n \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This can be seen as the tensors on the outer edge of the wedge do not have the operator pushed onto their outer indices.

This recursion is solved by

$$\begin{bmatrix} f_n \\ g_n \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \sum_{i=0}^{n-2} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}^{i} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The number of qudits needed to construct the operator is $N_{best} = 6f_n + 5g_n - 4$. Numerical results show the fraction of bulk qudits needed to reconstruct the center operator is

$$\lim_{n\to\infty} \frac{N_{best}}{N_{boundary}} \approx .467$$

Conclusion

It was shown explicitly how to construct perfect tensors that can then be used for holographic quantum codes. The fact that there always exists a perfect tensor with any indices given a large enough local dimension allows a variety of holographic codes to be constructed. These perfect tensors can be used in tensor networks based on certain hyperbolic tilings in order to create toy models of AdS/CFT. In the heptagon code, we found the minimal support of operators corresponding to center qudits to be relatively close to the AdS/CFT correspondence value of one half The heptagon code can protect from more erasures than the pentagon code in [1]. Unfortunately the heptagon code requires much more overhead as the rate is much less than that of the pentagon code. This is expected as the fraction of the indices needed to support an incoming operator is less than that for a six index perfect tensor. Thus when pushing operators to the boundary one can pick a narrower wedge than that of the pentagon code.

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