A Partial MaxSAT Approach to Nonmonotonic Reasoning with System W

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Abstract

The only recently introduced System W is a nonmonotonic inductive inference operator exhibiting some notable properties like extending rational closure and satisfying syntax splitting postulates for inference from conditional belief bases. A semantic model of system W is given by its underlying preferred structure of worlds, a strict partial order on the set of propositional interpretations, also called possible worlds, over the signature of the belief base. Existing implementations of system W are severely limited by the number of propositional variables that occur in a belief base because of the exponentially growing number of possible worlds. In this paper, we present an approach to realizing nonmonotonic reasoning with system W by using partial maximum satisfiability (PMaxSAT) problems and exploiting the power of current PMaxSAT solvers. An evaluation of our approach demonstrates that it outperforms previous implementations of system W and scales reasoning with system W up to a new dimension.

1 Introduction

Conditionals of the form "If *A* then usually *B*" establish a plausible connection between the antecedent *A* and the consequent *B*, while still allowing for exceptions. Such conditionals play a major role in uncertain reasoning, and many different semantics have been proposed for them, like probability distributions, plausibility orderings, possibility distributions, ranking functions and special instances of them, or conditional objects, (see, e.g., (Adams 1975; Nute 1980; Spohn 1988; Kraus, Lehmann, and Magidor 1990; Dubois and Prade 1994; Benferhat, Dubois, and Prade 1997; Kern-Isberner 2001; Beierle et al. 2021)).

Here, we consider system W (Komo and Beierle 2020; 2022), an only recently introduced inference method that exhibits some notable properties put forward as desirable for nonmonotonic reasoning from conditional belief bases. For instance, system W satisfies system P (Kraus, Lehmann, and Magidor 1990), extends rational closure (Lehmann and Magidor 1992), avoids the drowning problem (Benferhat et al. 1993), fully complies with syntax splitting (Kern-Isberner, Beierle, and Brewka 2020; Haldimann and Beierle 2022a; 2022b), and besides lexicographic inference (Lehmann 1995), up to now, system W is the only inference operator shown to satisfy also the more general property of conditional syntax splitting (Heyninck et al. 2023). However, the existing implementation of system W (Beierle et al. 2022) severely limits the number of propositional variables that can occur in a conditional belief base Δ. This is due to the exponentially growing number of possible worlds because the semantic model of system W, a strict partial order on the set of possible worlds, is constructed completely for computing a system W inference in the context of Δ.

In this paper, we present an approach to realizing nonmonotonic reasoning with system W by using partial maximum satisfiability (PMaxSAT) problems and exploiting the power of current PMaxSAT solvers (Larrosa and Rollon 2020; Bjørner et al. 2019; Ignatiev, Morgado, and Marques-Silva 2019). First evaluation results of our approach demonstrate that it outperforms previous implementations of system W and scales reasoning with system W up to a new dimension. In summary, the main contributions of this paper are:

- Modelling system W concepts as Partial MaxSAT problems.
- Development of SWinf, an algorithm for system W inference based on Partial MaxSAT.
- Implementation of SWinf in Python using a current MaxSAT solver.
- Evaluation of the SWinf implementation demonstrating its superiority over previous implementations.

After recalling the required background of conditional logic in Sec. 2, we briefly present system W in Sec. 3 and illustrate it with examples. In Sec. 4, we model system W concepts as Partial MaxSAT problems which are then used for developing SWinf in Sec. 5. Section 6 describes our implementation of SWinf and presents the resulting evaluation results. Section 7 concludes and points out future work.

2 Background: Conditional Logic

A *(propositional) signature* is a finite set Σ of propositional variables. For a signature Σ, we denote the propositional language over Σ by $\mathcal{L}_\Sigma$. Usually, we denote elements of signatures with lowercase letters $a, b, c, \ldots$ and formulas with uppercase letters $A, B, C, \ldots$. We may denote a conjunction $A \wedge B$ by AB and a negation $\neg A$ by $\overline{A}$ for brevity of notation. The set of interpretations over a signature Σ is denoted as Ω_Σ. Interpretations are also called *worlds* and Ω_Σ is called the *universe*. An interpretation $\omega \in \Omega_\Sigma$ is a *model* of a formula $A \in \mathcal{L}_\Sigma$ if A holds in ω. This is denoted as $\omega \models A$. The set of models of a formula (over a signature Σ) is denoted as $\mathit{Mod}_\Sigma(A) = \{\omega \in \Omega_\Sigma \mid \omega \models A\}$ or sometimes as Ω_A. The Σ in $\mathit{Mod}_\Sigma(A)$ can be omitted if the signature is clear from the context. A formula A *entails* a formula B if $\mathit{Mod}_\Sigma(A) \subseteq \mathit{Mod}_\Sigma(B)$. By slight abuse of notation we sometimes interpret worlds as the corresponding complete conjunction of all elements in the signature in either positive or negated form.

A *conditional* $(B|A)$ connects two formulas A, B and represents the rule "If A then usually B", where A is the *antecedent* and B the *consequent* of the conditonal. The conditional language over a signature Σ is denoted as $(\mathcal{L}|\mathcal{L})_\Sigma = \{(B|A) \mid A, B \in \mathcal{L}_\Sigma\}$. A *conditional belief base* is a finite set of conditionals. We use a three-valued semantics of conditionals in this paper (de Finetti 1937). For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. Popular models for conditional belief bases are ranking functions (also called ordinal conditional functions, OCF) (Spohn 1988) and total preorders (TPO) on Ω_Σ (Darwiche and Pearl 1997); transformations among these and other semantics are studied by Beierle and Kern-Isberner (2012). Semantic structures for conditionals have in common that they model a conditional $(B|A)$ if they consider its verification AB to be strictly more plausible, or less suprising, etc., than its falsification $A\overline{B}$; they model a belief base Δ if they model every conditional in Δ. A belief base Δ is *consistent* if it has a model.

Reasoning with conditionals is often modelled by inference relations. An *inference relation* is a binary relation $\mid\!\sim$ on formulas over an underlying signature Σ with the intuition that $A \mid\!\sim B$ means that A (plausibly) entails B. (Non-monotonic) inference is closely related to conditionals: an inference relation $\mid\!\sim$ can also be seen as a set of conditionals $\{(B|A) \mid A, B \in \mathcal{L}_\Sigma, A \mid\!\sim B\}$. An *inductive inference operator* (Kern-Isberner, Beierle, and Brewka 2020) is a function that maps each belief base to an inference relation.

3 System W

The inductive inference operator system W (Komo and Beierle 2020; 2022) takes into account the *tolerance information* expressed by the ordered partition of Δ.

Definition 1 (ordered partition $OP(\Delta) = (\Delta^0, \ldots, \Delta^k)$ (Goldszmidt and Pearl 1996))**.** *A conditional $(B|A)$ is* tolerated *by a set of conditionals Δ if there exists a world $\omega \in \Omega_\Sigma$ such that ω verifies $(B|A)$ and ω does not falsify any conditional in Δ, i.e., $\omega \models AB$ and $\omega \models \bigwedge_{i=1}^n (\overline{A_i} \vee B_i)$.*

The ordered partition $OP(\Delta) = (\Delta^0, \ldots, \Delta^k)$ *of a belief base Δ, also called tolerance partition, is the partition of Δ where each Δ_i is the (with respect to set inclusion) maximal subset of $\bigcup_{j=i}^k \Delta_j$ that is tolerated by $\bigcup_{j=i}^k \Delta_j$.*

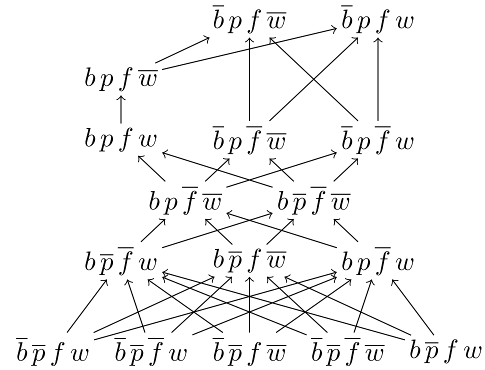

Figure 1: The preferred structure on worlds $<^{\mathsf{w}}_{\Delta_{bird}}$ induced by the belief base Δ_{bird} from Example 1.

It is well-known that $OP(\Delta)$ exists iff Δ is consistent (Pearl 1990).

Example 1 (Δ_{bird})**.** *Let $\Sigma = \{b, p, f, w\}$ be the signature representing birds, penguins, flying things and winged things, and let Δ_{bird} contain $r_1 = (f|b)$, $r_2 = (\overline{f}|p)$, $r_3 = (b|p)$, and $r_4 = (w|b)$. For instance, r_1 expresses* "birds usually fly"*. Δ_{bird} is consistent, and $OP(\Delta_{bird}) = (\Delta^0, \Delta^1)$ with $\Delta^0 = \{(f|b), (w|b)\}$ and $\Delta^1 = \{(\overline{f}|p), (b|p)\}$.*

In addition to the tolerance partion, system W also takes into account the structural information about which conditionals are falsified by a world, yielding the preferred structure on worlds.

Definition 2 (ξ^j, ξ, preferred structure $<^{\mathsf{w}}_\Delta$ on worlds (Komo and Beierle 2022))**.** *Consider a consistent belief base $\Delta = \{r_i = (B_i|A_i) \mid i \in \{1, \ldots, n\}\}$ with $OP(\Delta) = (\Delta^0, \ldots, \Delta^k)$. For $j = 0, \ldots, k$, the functions ξ^j and ξ are the* functions mapping worlds to the set of falsified conditionals *from the set Δ^j in the tolerance partition and from Δ, respectively, given by*

$$\xi^j(\omega) := \{r_i \in \Delta^j \mid \omega \models A_i\overline{B_i}\}, \tag{1}$$

$$\xi(\omega) := \{r_i \in \Delta \mid \omega \models A_i\overline{B_i}\}. \tag{2}$$

The preferred structure on worlds *is given by the binary relation $<^{\mathsf{w}}_\Delta \subseteq \Omega \times \Omega$ defined by, for any $\omega, \omega' \in \Omega$,*

$$\omega <^{\mathsf{w}}_\Delta \omega' \;\text{ iff }\; \text{there exists } m \in \{0, \ldots, k\} \text{ such that}$$
$$\xi^i(\omega) = \xi^i(\omega') \quad \forall i \in \{m+1, \ldots, k\}, \text{ and}$$
$$\xi^m(\omega) \subsetneqq \xi^m(\omega'). \tag{3}$$

Thus, $\omega <^{\mathsf{w}}_\Delta \omega'$ if and only if ω falsifies strictly fewer conditionals than ω' in the partition with the biggest index m where the conditionals falsified by ω and ω' differ. Note that $<^{\mathsf{w}}_\Delta$ is a strict partial order.

Example 2. *The preferred structure of worlds $<^{\mathsf{w}}_{\Delta_{bird}}$ for Δ_{bird} given in Example 1 is shown in Figure 1.*

The inductive inference operator system W is based on $<^{\mathsf{w}}_\Delta$ and is defined as follows.

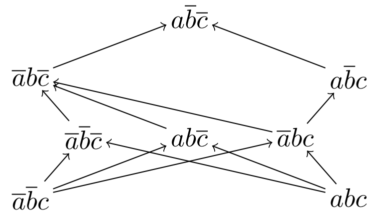

Figure 2: The preferred structure on worlds induced by the belief base Δ from Example 3.

Definition 3 (system W, $\mid\!\sim^{\mathsf{w}}_{\Delta}$(Komo and Beierle 2022))**.** *Let Δ be a belief base and A, B be formulas. Then B is a* system W inference from A (in the context of Δ)*, denoted $A \mid\!\sim^{\mathsf{w}}_{\Delta} B$, if we have:*

$$A \mid\!\sim^{\mathsf{w}}_{\Delta} B \;\; \textit{iff} \;\; \textit{for every } \omega' \in \Omega_{A\overline{B}} \; \textit{there is an } \omega \in \Omega_{AB} \textit{ s.t. } \omega <^{\mathsf{w}}_{\Delta} \omega' \tag{4}$$

Using $<^{\mathsf{w}}_{\Delta_{bird}}$ shown in Figure 1, we can check that $p \mid\!\sim^{\mathsf{w}}_{\Delta_{bird}} w$ holds, i.e., that penguins usually have wings is a system W inference in the context of Δ_{bird}. The following example illustrates system W in the case where the ordered partition of Δ is trivial.

Example 3. *Consider the belief base $\Delta = \{(b|a), (\overline{a}\overline{b}|\overline{a} \vee \overline{b}), (c|\top)\}$ over the signature $\Sigma = \{a, b, c\}$. Every conditional in Δ is tolerated by Δ, hence $OP(\Delta) = \{\Delta\}$. The preferred structure $<^{\mathsf{w}}_{\Delta}$ on worlds is given in Figure 2; note that $<^{\mathsf{w}}_{\Delta}$ is not a total preorder, and thus, it cannot be expressed by system Z nor by any other ranking model of Δ.*

Let us consider the question whether from $\overline{a}b \vee a\overline{b}$ we can infer $\overline{a}b$ in the context of Δ. This inference can not be obtained with p-entailment and neither with system Z. However, using the preferred structure $<^{\mathsf{w}}_{\Delta}$ given in Figure 2, it is straightforward to verify that for each world ω' with $\omega' \models a\overline{b}$ there is a world ω with $\omega \models \overline{a}b$ such that $\omega <^{\mathsf{w}}_{\Delta} \omega'$. Thus, with system W we obtain the inference $\overline{a}b \vee a\overline{b} \mid\!\sim^{\mathsf{w}}_{\Delta} \overline{a}b$.

4 System W and Partial MaxSAT Problems

For modelling system W with Partial MaxSAT, we generalize some of the notation introduced above. For a formula F and a set of formulas M over Σ, we use the following notations:

$$\omega \models M \;\text{ iff }\; \omega \models F_i \text{ for every } F_i \in M$$

$$\Omega_M = \{\omega \in \Omega_\Sigma \mid \omega \models M\}$$

$$\overline{M} = \{\overline{F}_i \mid F_i \in M\}$$

$$F \wedge M = F \wedge F_1 \wedge \ldots \wedge F_M \;\text{ for } M = \{F_1, \ldots, F_m\}$$

For a conditional $(B|A)$, the formula $\overline{A} \vee B$ expressing its *n*on-*f*alsification is denoted by $nf(B|A)$, and nf is extended canonically to a set Δ of condtionals. Thus

$$nf(\Delta) = \{\overline{A} \vee B \mid (B|A) \in \Delta\}$$

$$\overline{nf(\Delta)} = \{A\overline{B} \mid (B|A) \in \Delta\}$$

are the sets of non-falsifying and falsifying formulas, respectively, for the conditionals in Δ. The function ξ (Definition 2) is extended to sets of worlds $\Omega' \subseteq \Omega_\Sigma$ by defining $\xi(\Omega') = \{\xi(\omega') \mid \omega' \in \Omega'\}$. Furthermore, for a set M and a partial order $<$ on M, the minimal elements of $N \subseteq M$ are denoted by:

$$\min(N, <) = \{n \in N \mid \text{there is no } n' \in N \text{ s.t. } n' < n\}.$$

Because $<^{\mathsf{w}}_{\Delta}$ is a strict partial order, Definition 3 directly implies that is suffices to consider only the minimal worlds with respect the preferred structure of worlds for checking whether $\mid\!\sim^{\mathsf{w}}_{\Delta}$ holds.

Proposition 1 ($\mid\!\sim^{\mathsf{w}}_{\Delta}$)**.** *Let Δ be a belief base and $A, B \in \mathcal{L}$.*

$$A \mid\!\sim^{\mathsf{w}}_{\Delta} B \;\; \textit{iff} \;\; \textit{for every } \omega' \in \min(\Omega_{A\overline{B}}, <^{\mathsf{w}}_{\Delta}) \; \textit{there is an } \omega \in \min(\Omega_{AB}, <^{\mathsf{w}}_{\Delta}) \textit{ s.t. } \omega <^{\mathsf{w}}_{\Delta} \omega' \tag{5}$$

For computing min-expressions as they occur in Equation (5), we will employ MaxSAT concepts (Larrosa and Rollon 2020). Given a set of formulas S of soft constraints and a set of formulas H of hard constraints the *extended partial maximum satisfiability problem $EPMaxSAT(S, H)$* is the optimization problem of maximizing the number of satisfied formulas in S over all interpretations $\omega \in \Omega_H$ and determining all subsets of S that are maximal with this property.

Definition 4 ($MSS(S, H)$, $MCS(S, H)$)**.** *Let $S = \{S_1, \ldots, S_s\} \subseteq \mathcal{L}$ be a set of formulas, called* soft constraints*, and let $H = \{H_1, \ldots, H_h\} \subseteq \mathcal{L}$ be a set of formulas, called* hard constraints.

A set $M \subseteq S$ such that there is an $\omega \in \Omega_H$ with $\omega \models M$ and for every $M' \subseteq S$ with $M \subsetneq M'$ there is no $\omega' \in \Omega_H$ with $\omega' \models M'$ is called a maximal satisfiable subset (MSS) *with respect to (S, H), and $MSS(S, H)$ denotes the set of all MSS w.r.t. (S, H).*

A set $N \subseteq S$ is called a minimal correction subset (MCS) *with respect to (S, H) if $S \setminus N$ is an MSS w.r.t. (S, H), and $MCS(S, H)$ denotes the set of all MCS w.r.t. (S, H).*

Example 4. *Let Δ_{bird} and $OP(\Delta_{bird}) = (\Delta^0, \Delta^1)$ as in Example 1. For $S = nf(\Delta^1)$ and $H = \{pw\}$ we get*

$$MSS(nf(\Delta^1), \{pw\}) = MSS(\{\overline{p} \vee \overline{f}, \overline{p} \vee b\}, \{pw\}) = \{\{\overline{p} \vee \overline{f}, \overline{p} \vee b\}\}$$

and thus $MCS(nf(\Delta^1), \{pw\}) = \{\emptyset\}$.

For $S = nf(\Delta^0)$ and $H = nf(\Delta^1) \cup \{pw\}$ we get

$$MSS(nf(\Delta^0), nf(\Delta^1) \cup \{pw\}) =$$
$$MSS(\{\overline{b} \vee f, \overline{b} \vee w\}, \{\overline{p} \vee \overline{f}, \overline{p} \vee b, pw\}) = \;\; \{\{\overline{b} \vee w\}\}$$

and thus $MCS(nf(\Delta^0), nf(\Delta^1) \cup \{pw\}) = \{\{\overline{b} \vee f\}\}$.

In addition to using the minima as in Equation (5) for computing $\mid\!\sim^{\mathsf{w}}_{\Delta}$, we will exploit the fact that the underlying relation $\omega <^{\mathsf{w}}_{\Delta} \omega'$ (cf. Equation (2)) can be determined by going from the the highest partition element Δ^k in $OP(\Delta) = (\Delta^0, \ldots, \Delta^k)$ down to a lower element Δ^j only in case that $\xi^l(\omega) = \xi^l(\omega')$ for all $l \in \{j+1, \ldots, k\}$. For formalizing this observation, we introduce the following notion.

Algorithm 1 $\mathsf{SWinf}(\Delta, A, B)$

Input: belief base Δ and formulas A, B
Output: *Yes* if $A \mathrel{|\!\sim}^{\mathsf{w}}_{\Delta} B$, and *No* otherwise

1: **let** $OP(\Delta) = (\Delta^0, \dots, \Delta^k)$
2: **function** *recWinf*(j, H)
3: $\quad \mathcal{V} \leftarrow MCS(nf(\Delta^j), H \cup \{AB\})$
4: $\quad \mathcal{F} \leftarrow MCS(nf(\Delta^j), H \cup \{A\overline{B}\})$
5: $\quad$ **if** $\neg(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N')$ **then**
6: $\qquad$ **return** *No*
7: $\quad$ **for all** $N \in \mathcal{V} \cap \mathcal{F}$ **do**
8: $\qquad$ **if** $j = 0$ **then**
9: $\qquad\quad$ **return** *No*
10: $\qquad H_{new} \leftarrow (nf(\Delta^j) \setminus N) \cup \overline{N}$
11: $\qquad$ **if** *recWinf*$(j-1, H \cup H_{new})$ = *No* **then**
12: $\qquad\quad$ **return** *No*
13: $\quad$ **return** *Yes*
14: **end function**
15: **return** *recWinf*$(k, \emptyset)$

Definition 5 (nf/f-condition for (Δ, j))**.** *Let Δ be a belief base with $OP(\Delta) = (\Delta^0, \dots, \Delta^k)$, and let $j \in \{0, \dots, k\}$. A set of formulas H is a non-falsifying/falsifying condition (nf/f-condition) for (Δ, j) if there are, for $i \in \{j+1, \dots, k\}$ sets $\Delta^i_{nf}, \Delta^i_f \subseteq \Delta^i$ such that $\Delta^i = \Delta^i_{nf} \cup \Delta^i_f$ and $\Delta^i_{nf} \cap \Delta^i_f = \emptyset$, and*

$$H = \bigcup_{i \in \{j+1, \dots, k\}} nf(\Delta^i_{nf}) \cup \overline{nf(\Delta^i_f)}$$

Thus, given $OP(\Delta) = (\Delta^0, \dots, \Delta^k)$, an nf/f-condition H for (Δ, j) contains either the non-falsifying formula $\overline{A} \vee B$ or the falsifying formula $A\overline{B}$ for every conditional $(B|A) \in \Delta^{j+1} \cup \ldots \cup \Delta^k$; this way the nf/f-condition H ensures that for any two worlds $\omega, \omega' \in \Omega_\Sigma$ with $\omega \models H$ and $\omega' \models H$ we have $\xi^l(\omega) = \xi^l(\omega')$ for all $l \in \{j+1, \dots, k\}$. Note that H is an nf/f-condition for (Δ, k) iff $H = \emptyset$.

In the following section, we will present an algorithm using the MaxSat concepts presented above for computing system W inference.

5 The Algorithm SWinf

The algorithm $\mathsf{SWinf}(\Delta, A, B)$ (system W inference with Partial MaxSAT, Algorithm 1) takes a belief base Δ and two formulas A, B as input and answers the question whether $A \mathrel{|\!\sim}^{\mathsf{w}}_{\Delta} B$ holds.

The main part of SWinf is the recursive function *recWinf* (Lines 2 – 14) taking an index j pointing to the partition element Δ^j of $OP(\Delta)$ and an nf/f-condition for (Δ, j) as input. The inital call of *recWinf* is *recWinf*$(k, \emptyset)$ in Line 15.

In Line 3, *recWinf* assigns to $\mathcal{V}$ the minimal correcting subsets with respect to the set of non-falsifying formulas for the conditionals in the partition element Δ^j of $OP(\Delta)$ as soft constraints and the nf/f-condition for (Δ, j) obtained so far enlarged by the verification formula AB for the given query as hard constarints. Analogously, in Line 4, $\mathcal{F}$ is set to the minimal correcting subsets with respect to the same set of soft constraints and the set of hard constraints obtained by enlarging the nf/f-condition for (Δ, j) by the falsification formula $A\overline{B}$ for the given query.

If the condition in Line 5 holds, Equation (4) in the definition of $\mathrel{|\!\sim}^{\mathsf{w}}_{\Delta}$ is not satisfied, causing *recWinf* and thus also SWinf to return *No*.

If $\mathcal{V} \cap \mathcal{F}$ is empty (Line 7), *recWinf* returns *Yes*. Otherwsie, if $\mathcal{V} \cap \mathcal{F}$ is not empty, the next lower partion element of $OP(\Delta)$ has to be taken into account. If we are already at the lowest partition element, *recWinf* and also SWinf returns *No* (Line 8 –9). Otherwise, for every set N in $\mathcal{V} \cap \mathcal{F}$, a new nf/f-condition for $(\Delta, j-1)$ is constructed (Lines 10 – 11). SWinf will return *Yes* only if all resulting *recWinf* calls return *Yes*, otherwise SWinf returns *No*.

In the following, we will illustrate the execution of SWinf with several examples.

Example 5. *With Δ_{bird} from Example 1, executing $\mathsf{SWinf}(\Delta_{bird}, p, w)$ results in two succesive calls of recWinf involving the following values and conditions, cf. Example 4:*

recWinf$(j = 1, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\emptyset\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = true$
$\mathcal{V} \cap \mathcal{F} = \{\emptyset\}$
$j = 1 > 0$
return Yes *iff recWinf*$(0, \{\overline{p} \vee \overline{f}, \overline{p} \vee b\})$ = Yes

recWinf$(j = 0, H = \{\overline{p} \vee \overline{f}, \overline{p} \vee b\})$
$\mathcal{V} = \{\{\overline{b} \vee f\}\}$
$\mathcal{F} = \{\{\overline{b} \vee f, \overline{b} \vee w\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = true$
$\mathcal{V} \cap \mathcal{F} = \emptyset \rightarrow$ *return* Yes

Thus, $\mathsf{SWinf}(\Delta_{bird}, p, w)$ *returns* Yes *and* $p \mathrel{|\!\sim}^{\mathsf{w}}_{\Delta_{bird}} w$. *When asking whether $\overline{w}$ can be inferred from p in the context of Δ_{bird} with system W,* $\mathsf{SWinf}(\Delta_{bird}, p, \overline{w})$ *yields* No*:*

recWinf$(j = 1, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\emptyset\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = true$
$\mathcal{V} \cap \mathcal{F} = \{\emptyset\}$
$j = 1 > 0$
return Yes *iff recWinf*$(0, \{\overline{p} \vee \overline{f}, \overline{p} \vee b\})$ = Yes

recWinf$(j = 0, H = \{\overline{p} \vee f, \overline{p} \vee b\})$
$\mathcal{V} = \{\{\overline{b} \vee f, \overline{b} \vee w\}\}$
$\mathcal{F} = \{\{\overline{b} \vee f\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = false \rightarrow$ *return* No

Example 6. *With Δ from Example 3, executing* $\mathsf{SWinf}(\Delta, b, a)$ *yields* Yes*:*

recWinf$(j = 1, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\{ab \vee \overline{a}\overline{b}\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = true$
$\mathcal{V} \cap \mathcal{F} = \emptyset \rightarrow$ *return* Yes

Executing $\mathsf{SWinf}(\Delta, c, a)$ *triggers the condition in Line 8 and yields* No*:*

recWinf$(j = 1, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\emptyset\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \{\emptyset\}$
$j = 0 \rightarrow$ *return* No

Example 7. *Let* $\Delta = \{(B_1|A_1), \ldots, (B_9|A_9)\}$ *be a belief base with nine conditionals and the ordered partition:*

$$\begin{aligned} &OP(\Delta) = (\Delta^0, \Delta^1, \Delta^2) \\ &\quad \Delta_0 = \{\{(B_1|A_1), (B_2|A_2), (B_3|A_3), (B_4|A_4)\} \\ &\quad \Delta_1 = \{(B_5|A_5), (B_6|A_6), (B_7|A_7)\} \\ &\quad \Delta_2 = \{(B_8|A_8), (B_9|A_9)\} \end{aligned}$$

Query 7.1. *Assume that for* A, B *the following holds:*

$$\begin{aligned} \xi(\min(\Omega_{AB}, <^{\mathsf{w}}_{\Delta})) = \{&\{(B_1|A_1), (B_5|A_5)\}, \\ &\{(B_1|A_1), (B_7|A_7)\}, \\ &\{(B_4|A_4), (B_7|A_7)\}\} \\ \xi(\min(\Omega_{A\overline{B}}, <^{\mathsf{w}}_{\Delta})) = \{&\{(B_5|A_5), (B_6|A_6)\}, \\ &\{(B_1|A_1), (B_2|A_2), (B_7|A_7)\}, \\ &\{(B_3|A_3), (B_4|A_4), (B_7|A_7)\}\} \end{aligned}$$

Then executing $\mathsf{SWinf}(\Delta, A, B)$ *produces the following values and conditions for the first call of recWinf:*

recWinf$(j = 2, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\emptyset\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \{\emptyset\}$
$j = 2 > 0$
return Yes iff recWinf$(1, \{\overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\}) =$ Yes

The recursive call of recWinf yields:

recWinf$(j = 1, H = \{\overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\})$
$\mathcal{V} = \{\{\overline{A}_5 \vee B_5\}, \{\overline{A}_7 \vee B_7\}\}$
$\mathcal{F} = \{\{\overline{A}_5 \vee B_5, \overline{A}_6 \vee B_6\}, \{\overline{A}_7 \vee B_7\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \{\{\overline{A}_7 \vee B_7\}\}$
$j = 1 > 0$
return Yes iff recWinf$(0, \{\overline{A}_5 \vee B_5, \overline{A}_6 \vee B_6, A_7\overline{B}_7, \overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\}) =$ Yes

The new recursive call of recWinf then yields:

recWinf$(j = 0, H = \{\overline{A}_5 \vee B_5, \overline{A}_6 \vee B_6, A_7\overline{B}_7, \overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\})$
$\mathcal{V} = \{\{\overline{A}_1 \vee B_1\}, \{\overline{A}_4 \vee B_4\}\}$
$\mathcal{F} = \{\{\overline{A}_1 \vee B_1, \overline{A}_2 \vee B_2\}, \{\overline{A}_3 \vee B_3, \overline{A}_4 \vee B_4\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \emptyset \rightarrow$ *return* Yes

Since $\mathcal{V} \cap \mathcal{F} = \emptyset$*, the condition in the for-statement (Line 7) is trivially satisfied, and recWinf and also* $\mathsf{SWinf}(\Delta, A, B)$ *return* Yes*; thus,* $A \mathrel{|\!\sim}^{\mathsf{w}}_{\Delta} B$*.*

Query 7.2. *Assume that for* A', B' *the following holds:*

$$\begin{aligned} \xi(\min(\Omega_{A'B'}, <^{\mathsf{w}}_{\Delta})) = \{&\{(B_1|A_1), (B_5|A_5)\}, \\ &\{(B_4|A_4), (B_5|A_5)\}, \\ &\{(B_1|A_1), (B_7|A_7)\}, \\ &\{(B_4|A_4), (B_7|A_7)\}\} \\ \xi(\min(\Omega_{A'\overline{B'}}, <^{\mathsf{w}}_{\Delta})) = \{&\{(B_1|A_1), (B_5|A_5)\}, \\ &\{(B_3|A_3), (B_4|A_4), (B_5|A_5)\}, \\ &\{(B_1|A_1), (B_2|A_2), (B_7|A_7)\}, \\ &\{(B_3|A_3), (B_4|A_4), (B_7|A_7)\}\} \end{aligned}$$

Then executing $\mathsf{SWinf}(\Delta, A', B')$ *produces the following values and conditions for the first call of recWinf:*

recWinf$(j = 2, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\emptyset\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \{\emptyset\}$
$j = 2 > 0$
return Yes *iff recWinf*$(1, \{\overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\}) =$ Yes

This new call of recWinf causes two further calls of recWinf, the first one containing the falisification for $(B_5|A_5)$*, the second one containing the falisification for* $(B_7|A_7)$ *among its hard constraints:*

recWinf$(j = 1, \{\overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\})$
$\mathcal{V} = \{\{\overline{A}_5 \vee B_5\}, \{\overline{A}_7 \vee B_7\}\}$
$\mathcal{F} = \{\{\overline{A}_5 \vee B_5\}, \{\overline{A}_7 \vee B_7\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \{\{\overline{A}_5 \vee B_5\}, \{\overline{A}_7 \vee B_7\}\}$
$j = 1 > 0$
return Yes *iff*
(*recWinf*$(0, \{A_5\overline{B}_5, \overline{A}_6 \vee B_6, \overline{A}_7 \vee B_7, \overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\}) =$ Yes$) \wedge$
(*recWinf*$(0, \{\overline{A}_5 \vee B_5, \overline{A}_6 \vee B_6, A_7\overline{B}_7, \overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\}) =$ Yes)

The first one of these two new calls of recWinf yields No*, the second one yields* Yes*:*

recWinf$(j = 0, H = \{A_5\overline{B}_5, \overline{A}_6 \vee B_6, \overline{A}_7 \vee B_7, \overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\})$
$\mathcal{V} = \{\{\overline{A}_1 \vee B_1\}, \{\overline{A}_4 \vee B_4\}\}$
$\mathcal{F} = \{\{\overline{A}_1 \vee B_1\}, \{\overline{A}_3 \vee B_3, \overline{A}_4 \vee B_4\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \{\{\overline{A}_1 \vee B_1\}\}$
$j = 0 \rightarrow$ *return* No

recWinf$(j = 0, H = \{\overline{A}_5 \vee B_5, \overline{A}_6 \vee B_6, A_7\overline{B}_7, \overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\})$
$\mathcal{V} = \{\{\overline{A}_1 \vee B_1\}, \{\overline{A}_4 \vee B_4\}\}$
$\mathcal{F} = \{\{\overline{A}_1 \vee B_1, \overline{A}_2 \vee B_2\}, \{\overline{A}_3 \vee B_3, \overline{A}_4 \vee B_4\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') =$ *true*
$\mathcal{V} \cap \mathcal{F} = \emptyset \rightarrow$ *return* Yes

Because not all of the recursive calls of recWinf return Yes*,* $\mathsf{SWinf}(\Delta, A', B')$ *returns* No*; thus,* $A' \not\mathrel{|\!\sim}^{\mathsf{w}}_{\Delta} B'$*.*

$\|\Delta\|$	6	8	10	12	14	16	18	20	30	40	50	60
$\|\Sigma\|$	6	8	10	12	14	16	18	20	30	40	50	60
WJ	19	204	4967	248323	timeout	timeout	timeout	timeout	timeout	timeout	timeout	timeout
SWinf	32	42	50	57	62	69	74	82	116	151	205	226

Table 1: Evaluation comparing different implementations of system W inference, time in milliseconds. Timeout was at 30 min.

Query 7.3. *Assume that for A'', B'' the following holds:*

$$\xi(\min(\Omega_{A''B''}, <^{\mathsf{w}}_{\Delta})) = \{\{(B_5|A_5), (B_6|A_6)\}, \{(B_7|A_7)\}\}$$
$$\xi(\min(\Omega_{A''\overline{B''}}, <^{\mathsf{w}}_{\Delta})) = \{\{(B_5|A_5)\}, \{(B_7|A_7)\}\}$$

Then executing $\mathsf{SWinf}(\Delta, A'', B'')$ *produces the following values and conditions for the first call of recWinf:*

$recWinf(j = 2, H = \emptyset)$
$\mathcal{V} = \{\emptyset\}$
$\mathcal{F} = \{\emptyset\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = true$
$\mathcal{V} \cap \mathcal{F} = \{\emptyset\}$
$j = 2 > 0$
return Yes *iff* $recWinf(1, \{\overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\}) =$ Yes

This new call of recWinf returns No *as follows:*

$recWinf(j = 1, H = \{\overline{A}_8 \vee B_8, \overline{A}_9 \vee B_9\})$
$\mathcal{V} = \{\{\overline{A}_5 \vee B_5, \overline{A}_6 \vee B_5\}, \{\overline{A}_7 \vee B_7\}\}$
$\mathcal{F} = \{\{\overline{A}_5 \vee B_5\}, \{\overline{A}_7 \vee B_7\}\}$
$(\forall N' \in \mathcal{F}\, \exists N \in \mathcal{V} \,.\, N \subseteq N') = false \rightarrow$ *return* No

$\mathsf{SWinf}(\Delta, A'', B'')$ *returns* No*; thus,* $A'' \not\mid\!\sim^{\mathsf{w}}_{\Delta} B''$.

6 Implementation and Evaluation Results

We implemented the algorithm SWinf in Python und using the MaxSAT solver Z3 (Bjørner et al. 2019). In our implementation, we used the optimizing features of the Z3 SMT Solver to find Pareto fronts (Bjørner, Phan, and Fleckenstein 2015) (which, in our case, is equivalent to finding the sets of all MSS) of non-falsifying formulas of the conditionals of a partition. We then derived our sets of all MCS due to the complementary nature of MSS and MCS.

We evaluated our implementation against an existing one using a different approach which is implemented in Java and creates a graph over all possible worlds to compare verifying and falsifying worlds with respect to a query (Beierle et al. 2022). The implementations were evaluated with a set of belief bases of increasing sizes and corresponding queries (available in the CLKR online repository (Beierle, Haldimann, and Schwarzer 2024) at `https://www.fernuni-hagen.de/wbs/clkr`). The belief bases Δ and queries were generated through a randomized process involving a signature Σ, and the size of the belief bases $|\Delta|$ was chosen to match the number of signature elements $|\Sigma|$ in each example. To avoid trivial cases, only consistent belief bases were taken into account. Keep in mind that the number of worlds which potentially have to be considered increases exponentially relative to the size of the signature. The evaluation was executed on an Intel i7-3770 CPU with 32GB DDR3-1666 working memory.

Table 1 summarizes the comparative evaluations of answering system W queries. Each column represents a choice of signature size and number of conditionals as specified in the first two rows. The third row starting with WJ corresponds to the results using the Java implementation described in (Beierle et al. 2022), and the fourth row gives the results of our Partial MaxSAT implementation of SWinf.

The times measured are given in milliseconds and represent the mean over 1000 different queries over 100 different belief bases of the respective sizes. The times include preprocessing needed by the specific implementations, i.e., building the graph representing the preferred structure of worlds induced by Δ for the Java-based implementation WJ, and computing the ordered partitioning $OP(\Delta)$ of Δ in the case of SWinf. We set a timeout of 30 minutes (1,800,000 ms) for each query. On the very smallest belief bases (6 signature elements and 6 conditionals), the implementation WJ outperforms our SWinf implementation. As soon as the belief bases grow in size, SWinf outperforms WJ, beating it by multiple orders of magnitude for belief bases with 12 signature elements and 12 conditionals. Additionally, the size of the signature and the number of conditionals in a belief base, which allow for a system W query to be answered within a reasonable time frame, have expanded considerably. While the WJ implementation could not answer queries for a belief base with 14 signature elements and 14 conditionals within 30 minutes, queries for belief bases with 60 signature elements and 60 conditionals, and thus involving 2^{60} possible worlds, can be answered within less than one second using our SWinf implementation.

7 Conclusions and Future Work

In this paper, we model system W concepts as Partial MaxSAT problems and use this for developing SWinf, an algorithm for system W inference that we implemented in Python and that uses a current MaxSAT solver. Our evaluation demonstrates that SWinf clearly outperforms previous implementations and scales up system W inference to a new dimension. Our future work includes further optimizing our approach, evaluating it with respect to different MaxSAT solvers, contrasting it to SAT/SMT based implementations of other inference methods for conditional belief bases, e.g., c-inference (Beierle et al. 2018; Beierle, von Berg, and Sanin 2022; von Berg, Sanin, and Beierle 2023), and integrating it in the InfOCF reasoning platform (Beierle, Eichhorn, and Kutsch 2017; Kutsch and Beierle 2021).

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References

Adams, E. 1975. *The Logic of Conditionals*. Dordrecht: D. Reidel.

Beierle, C., and Kern-Isberner, G. 2012. Semantical investigations into nonmonotonic and probabilistic logics. *Annals of Mathematics and Artificial Intelligence* 65(2-3):123–158.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2018. Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem. *Ann. Math. Artif. Intell.* 83(3-4):247–275.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2021. Properties and interrelationships of skeptical, weakly skeptical, and credulous inference induced by classes of minimal models. *Artificial Intelligence* 297:103489.

Beierle, C.; Haldimann, J.; Kollar, D.; Sauerwald, K.; and Schwarzer, L. 2022. An implementation of nonmonotonic reasoning with system W. In Bergmann, R.; Malburg, L.; Rodermund, S. C.; and Timm, I. J., eds., *KI 2022: Advances in Artificial Intelligence - 45th German Conference on AI, Trier, Germany, September 19-23, 2022, Proceedings*, volume 13404 of *LNCS*, 1–8. Springer.

Beierle, C.; Eichhorn, C.; and Kutsch, S. 2017. A practical comparison of qualitative inferences with preferred ranking models. *KI – Künstliche Intelligenz* 31(1):41–52.

Beierle, C.; Haldimann, J.; and Schwarzer, L. 2024. CLKR – Conditional Logic and Knowledge Representation. *KI – Künstliche Intelligenz*. (to appear).

Beierle, C.; von Berg, M.; and Sanin, A. 2022. Realization of c-inference as a SAT problem. *The International FLAIRS Conference Proceedings* 35.

Benferhat, S.; Cayrol, C.; Dubois, D.; Lang, J.; and Prade, H. 1993. Inconsistency Management and Prioritized Syntax-Based Entailment. In *Proc. IJCAI'93*, volume 1, 640–647. San Francisco, CA, USA: Morgan Kaufmann Publishers.

Benferhat, S.; Dubois, D.; and Prade, H. 1997. Nonmonotonic reasoning, conditional objects and possibility theory. *Artificial Intelligence* 92:259–276.

Bjørner, N.; de Moura, L.; Nachmanson, L.; and Wintersteiger, C. M. 2019. Programming Z3. *Engineering Trustworthy Software Systems: 4th International School, SETSS 2018, Chongqing, China, April 7–12, 2018, Tutorial Lectures 4* 148–201.

Bjørner, N. S.; Phan, A.; and Fleckenstein, L. 2015. νz - an optimizing SMT solver. In Baier, C., and Tinelli, C., eds., *Tools and Algorithms for the Construction and Analysis of Systems - 21st International Conference, TACAS 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015. Proceedings*, volume 9035 of *LNCS*, 194–199. Springer.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artif. Intell.* 89(1-2):1–29.

de Finetti, B. 1937. La prévision, ses lois logiques et ses sources subjectives. *Ann. Inst. H. Poincaré* 7(1):1–68. Engl. transl. *Theory of Probability*, J. Wiley & Sons, 1974.

Dubois, D., and Prade, H. 1994. Conditional objects as nonmonotonic consequence relationships. *Special Issue on Conditional Event Algebra, IEEE Transactions on Systems, Man and Cybernetics* 24(12):1724–1740.

Goldszmidt, M., and Pearl, J. 1996. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artificial Intelligence* 84(1-2):57–112.

Haldimann, J., and Beierle, C. 2022a. Inference with system W satisfies syntax splitting. In Kern-Isberner, G.; Lakemeyer, G.; and Meyer, T., eds., *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022, Haifa, Israel. July 31 - August 5, 2022*, 405–409.

Haldimann, J., and Beierle, C. 2022b. Properties of system W and its relationships to other inductive inference operators. In Varzinczak, I., ed., *Foundations of Information and Knowledge Systems - 12th International Symposium, FoIKS 2022, Helsinki, Finland, June 20-23, 2022, Proceedings*, volume 13388 of *LNCS*, 206–225. Springer.

Heyninck, J.; Kern-Isberner, G.; Meyer, T.; Haldimann, J. P.; and Beierle, C. 2023. Conditional syntax splitting for nonmonotonic inference operators. In Williams, B.; Chen, Y.; and Neville, J., eds., *Proceedings of the 37th AAAI Conference on Artificial Intelligence*, volume 37, 6416–6424.

Ignatiev, A.; Morgado, A.; and Marques-Silva, J. 2019. RC2: an efficient MaxSAT solver. *J. Satisf. Boolean Model. Comput.* 11(1):53–64.

Kern-Isberner, G.; Beierle, C.; and Brewka, G. 2020. Syntax splitting = relevance + independence: New postulates for nonmonotonic reasoning from conditional belief bases. In Calvanese, D.; Erdem, E.; and Thielscher, M., eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the 17th International Conference, KR 2020*, 560–571. IJCAI Organization.

Kern-Isberner, G. 2001. *Conditionals in nonmonotonic reasoning and belief revision*, volume 2087 of *LNAI*. Springer.

Komo, C., and Beierle, C. 2020. Nonmonotonic inferences with qualitative conditionals based on preferred structures on worlds. In Schmid, U.; Klügl, F.; and Wolter, D., eds., *KI 2020: Advances in Artificial Intelligence - 43rd German Conference on AI, Bamberg, Germany, September 21-25, 2020, Proceedings*, volume 12325 of *LNCS*, 102–115. Springer.

Komo, C., and Beierle, C. 2022. Nonmonotonic reasoning from conditional knowledge bases with system W. *Ann. Math. Artif. Intell.* 90(1):107–144.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intell.* 44(1-2):167–207.

Kutsch, S., and Beierle, C. 2021. InfOCF-Web: An online tool for nonmonotonic reasoning with conditionals and ranking functions. In Zhou, Z., ed., *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021*, 4996–4999. ijcai.org.

Larrosa, J., and Rollon, E. 2020. Towards a better understanding of (partial weighted) MaxSAT proof systems. In Pulina, L., and Seidl, M., eds., *Theory and Applications of Satisfiability Testing - SAT 2020 - 23rd International Conference, Alghero, Italy, July 3-10, 2020, Proceedings*, volume 12178 of *LNCS*, 218–232. Springer.

Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artif. Intell.* 55:1–60.

Lehmann, D. 1995. Another perspective on default reasoning. *Ann. Math. Artif. Intell.* 15(1):61–82.

Nute, D. 1980. *Topics in Conditional Logic*. Dordrecht, Holland: D. Reidel Publishing Company.

Pearl, J. 1990. System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *Proc. of the 3rd Conf. on Theoretical Aspects of Reasoning About Knowledge (TARK'1990)*, 121–135. San Francisco, CA, USA: Morgan Kaufmann Publ. Inc.

Spohn, W. 1988. Ordinal conditional functions: a dynamic theory of epistemic states. In Harper, W., and Skyrms, B., eds., *Causation in Decision, Belief Change, and Statistics, II*. Kluwer Academic Publishers. 105–134.

von Berg, M.; Sanin, A.; and Beierle, C. 2023. Representing nonmonotonic inference based on c-representations as an SMT problem. In Bouraoui, Z.; Jabbour, S.; and Vesic, S., eds., *Symbolic and Quantitative Approaches to Reasoning with Uncertainty - 17th European Conference, ECSQARU 2023, Arras, France, September 19–22, 2023, Proceedings*, volume 14249 of *LNCS*, 210–223. Springer.