Observational Equivalence of Conditional Belief Bases

Christoph Beierle, Jonas Haldimann, Leon Schwarzer
Knowledge-Based Systems, Faculty of Mathematics and Computer Science
FernUniversität in Hagen, 58084 Hagen, Germany
{christoph.beierle, jonas.haldimann, leon.schwarzer}@fernuni-hagen.de

Abstract

In nonmonotonic reasoning, a conditional of the form *'If A then usually B'* is typically accepted if a situation where both A and B hold is deemed to be more plausible, more probable, or less surprising, etc., than a situation where A holds, but B does not hold. In a propositional setting, this leads to a relation on the propositional interpretations, also called worlds, by comparing worlds according to their plausibility. In this paper, we address the question of which kind of relations on the set of worlds can be obtained by completing a conditional belief base via an inductive inference operator. As a key concept for our investigations, we introduce and employ the notion of observational equivalence of belief bases that takes an inductive inference operator and a set of queries into account. This leads to the notion of the observational normal form (ONF) and, by focussing on so-called base conditionals, to the base conditional normal form (BCNF). The acceptance of base conditionals corresponds to the plausibility ordering on possible worlds induced by an inference method. Both normal forms ONF and BCNF are combined with renamings as an additional dimension. We establish the interrelationships among the normal forms and evaluate them empirically with respect to systematically generated belief bases.

1 Introduction

Normal forms of syntactic representations occur in many areas of computer science. Benefits of normal forms can include, e.g., more succinct or standardized representations, simpler comparisons, avoidence of redundencies, and more efficient processing algorithms. While recently, normal forms of conditional belief bases, containing conditionals of the form *"If A then usually B"*, have gained some attention (e.g. (Beierle 2019; Beierle and Kutsch 2019a; 2019b; Beierle and Haldimann 2020a; 2020b; Beierle, Haldimann, and Kutsch 2021)), the main focus of research in conditional belief bases has been on different semantics (e.g., Lewis' system of spheres (Lewis 1973), conditional objects evaluated using boolean intervals (Dubois and Prade 1994), possibility distributions (Benferhat, Dubois, and Prade 1999), or ranking functions (Spohn 1988; 2012)) and on general inference properties (e.g. (Adams 1965; Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992)).

Important properties of normal forms like providing, possibly unique, representations for all relevant considered items, rely on both syntactic and semantic aspects. In (Beierle and Haldimann 2022a), normal forms of belief bases are introduced from the viewpoint obtained by respecting inference methods satisfying corresponding properties expressed by postulates put forward for nonmonotonic reasoning. In this paper, we extend this line of research by introducing the new notion of *observational equivalence* of conditional belief bases. In contrast to inferential equivalence, requiring that the complete inference relations induced by two belief bases Δ_1 and Δ_2 coincide, observational equivalence takes a set Q of queries into account. If Δ_1 answers the queries in Q precisely as Δ_2, then Δ_1 and Δ_2 are observationally equivalent. Use cases of such a scenario are situations where one is interested in the response behaviour with respect to only a subset of all possible queries, for instance in the framework of intentional forgetting when abstracting from details or when focussing on particular aspects, see e.g. (Eiter and Kern-Isberner 2019; Beierle et al. 2019). Observational equivalence that besides Q also depends on the considered inference method, leads to the *observational equivalence normal form (ONF)* and we show that for each Δ there is a corresponding, uniquely determined normal form.

A specific instance of observational equivalence, called *base conditional normal form (BCNF)*, characterizes precisely the situation where Δ_1 and Δ_2 agree on the relative plausibility of all possible worlds, i.e., Δ_1 considers a world ω strictly more plausible than a world ω' if and only if Δ_2 does so. Employment of the BCNF enables one to determine the strict partial orders on possible worlds that can be realized. Our empirical evaluations show, for instance, that with p-entailment it is possible to realize all 219 strict partial orders on the four worlds over a two-element signature, while for several other established inference methods, only a subset of these partial orders can be achieved. As an additional dimension, we study the effect of signature variable renamings, both of belief bases and of queries on observational equivalence and on the normal forms ONF and BCNF.

In summary, the main contributions of this paper are:

- observational equivalence of conditional belief bases with respect to an inductive inference operator and a set of queries,

- induced normal forms observational normal form (ONF) and base conditional normal form (BCNF),
- orthogonal combination with renamings of belief bases and of queries,
- empirical evaluations over systematically generated belief bases.

In Section 2 we present basics of conditional logic, and Section 3 recalls the required background of inductive inference operators and inferential equivalence. Section 4 introduces observational equivalence and ONF, Section 5 introduces base conditional equivalence and BCNF, and Section 6 introduces renamings on observations, while Section 7 concludes.

2 Background: Conditional Logic

Let $\mathcal{L}(\Sigma)$, or just $\mathcal{L}$, be the propositional language over a finite signature Σ. We call a signature Σ with a linear ordering $\lessdot$ an *ordered signature* and denote it by $(\Sigma, \lessdot)$. For $A, B \in \mathcal{L}$, we write AB for $A \wedge B$ and $\overline{A}$ for $\neg A$. We identify the set of all complete conjunctions over Σ with the set Ω of possible worlds over $\mathcal{L}$. For $\omega \in \Omega$ and $A \in \mathcal{L}$, $\omega \models A$ means that A holds in ω. The set of worlds satisfying A is $\Omega_A = \{\omega \mid \omega \models A\}$. Two formulas A, B are *equivalent*, denoted as $A \equiv B$, if $\Omega_A = \Omega_B$.

We define the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. The intuition of a conditional $(B|A)$ is that if A holds then usually B holds, too. As semantics for conditionals, we use functions $\kappa : \Omega \to \mathbb{N}$ such that $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$, called *ordinal conditional functions (OCF)*, introduced (in a more general form) by Spohn (1988). They express degrees of plausibility where a lower degree denotes "less surprising". Each κ uniquely extends to a function $\kappa : \mathcal{L} \to \mathbb{N} \cup \{\infty\}$ with $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$ where $\min \emptyset = \infty$. An OCF κ *accepts* a conditional $(B|A)$, written $\kappa \models (B|A)$, if $\kappa(AB) < \kappa(A\overline{B})$. A conditional $(B|A)$ is trivial if it is *self-fulfilling* ($A \models B$) or *contradictory* ($A \models \overline{B}$). We say that $(B|A)$ and $(B'|A')$ are *conditionally equivalent*, denoted by $(B|A) \equiv_{ce} (B'|A')$, if $A \equiv A'$ and $AB \equiv A'B'$. A finite set $\Delta \subseteq (\mathcal{L}|\mathcal{L})$ is a *belief base*. An OCF κ accepts Δ if κ accepts all conditionals in Δ, and Δ is *consistent* if an OCF accepting Δ exists.

For orderings like $\leqslant$ or $\preceq$ the strict variants are denoted by $<$ or $\prec$, respectively, i.e., $a < b$ iff $a \leqslant b$ and $b \nleqslant a$.

3 Inductive Inference, Normal Form Conditionals, and Inferential Equivalence

An *inference relation* $\mathrel{|\!\sim} \subseteq \mathcal{L} \times \mathcal{L}$ is a binary relation on formulas capturing (nonmonotonic) inferences: $A \mathrel{|\!\sim} B$ iff B can be inferred from A. The concept of an *inductive inference operator* formalizes how an inference relation $\mathrel{|\!\sim} \subseteq \mathcal{L} \times \mathcal{L}$ is obtained by inductive completion of a given belief base.

Definition 1 (inductive inference operator (Kern-Isberner, Beierle, and Brewka 2020))**.** *An* inductive inference operator *is a mapping* $C\colon \Delta \mapsto \mathrel{|\!\sim}_\Delta$ *that maps a belief base to an inference relation such that direct inference (DI) and trivial vacuity (TV) are fulfilled:*

(DI) *if* $(B|A) \in \Delta$ *then* $A \mathrel{|\!\sim}_\Delta B$

(TV) *if* $\Delta = \emptyset$ *and* $A \mathrel{|\!\sim}_\Delta B$ *then* $A \models B$

If no confusion arises, we will often simply use $\mathrel{|\!\sim}$ to denote the inductive inference operator mapping Δ to $\mathrel{|\!\sim}_\Delta$. Examples of inductive inference operators are:

p-entailment $\mathrel{|\!\sim}^p$ (Goldszmidt and Pearl 1996) considers all ranking models and coincides with system P-inference (Lehmann and Magidor 1992; Dubois and Prade 1994).

system Z $\mathrel{|\!\sim}^z$ (Goldszmidt and Pearl 1996) uses the inclusion maximal tolerance partition of Δ and it coincides with rational closure (Lehmann and Magidor 1992).

c-inference $\mathrel{|\!\sim}^c$ (Beierle et al. 2018; 2021) considers all c-representations (Kern-Isberner 2001; 2004).

system W $\mathrel{|\!\sim}^{\mathsf{w}}$ (Komo and Beierle 2020; 2022) captures both c-inference and system Z and thus rational closure.

In the following, we formalize some properties an inductive inference operator can have: (AND) and right weakening (RW) from system P, self-fulfilling (SF), semi-monotony (SM) (Reiter 1980; Goldszmidt and Pearl 1996), syntax independence (SI), and conditional equivalence (CE).

(AND) $A \mathrel{|\!\sim} B$ and $A \mathrel{|\!\sim} C$ imply $A \mathrel{|\!\sim} B \wedge C$

(RW) $B \models C$ and $A \mathrel{|\!\sim} B$ imply $A \mathrel{|\!\sim} C$

(SF) $A \models B$ implies $\mathrel{|\!\sim}_{\Delta \cup \{(B|A)\}} = \mathrel{|\!\sim}_\Delta$

(SM) $\Delta \subseteq \Delta'$ and $A \mathrel{|\!\sim}_\Delta B$ imply $A \mathrel{|\!\sim}_{\Delta'} B$

(SI) $A \equiv A'$ and $B \equiv B'$ imply $\mathrel{|\!\sim}_{\Delta \cup \{(B|A)\}} = \mathrel{|\!\sim}_{\Delta \cup \{(B'|A')\}}$

(CE) $(B|A) \equiv_{ce} (B'|A')$ imp. $\mathrel{|\!\sim}_{\Delta \cup \{(B|A)\}} = \mathrel{|\!\sim}_{\Delta \cup \{(B'|A')\}}$

For comparing belief bases, an important criterion is whether they induce the same entailments. In a nonmonotonic setting, this obviously depends on the inductive inference operator inducing the entailments.

Definition 2 ($\equiv_{\mathrel{|\!\sim}}$, inferentially equivalent with respect to $\mathrel{|\!\sim}$ (Beierle and Haldimann 2022a))**.** *Two belief bases* Δ, Δ' *are* inferentially equivalent with respect to $\mathrel{|\!\sim}$*, denoted by* $\Delta \equiv_{\mathrel{|\!\sim}} \Delta'$*, if, for all formulas* A, B*,* $A \mathrel{|\!\sim}_\Delta B$ *holds if and only if* $A \mathrel{|\!\sim}_{\Delta'} B$*.*

A desirable property for a normal form *<NF>* is that it also covers all sets of entailments that can be obtained from a belief base not in *<NF>*. In the following, we will use $\boldsymbol{\Delta}$(*<NF>*) to denote the set of all belief bases in *<NF>*.

Definition 3 ($\mathrel{|\!\sim}$-complete (Beierle and Haldimann 2022a))**.** *IR* $\subseteq \mathcal{L} \times \mathcal{L}$ *is a* $\mathrel{|\!\sim}$-relation *if there is a consistent belief base* Δ *with* $\mathrel{|\!\sim}_\Delta = IR$*. A set* $\mathcal{S}$ *of belief bases is* $\mathrel{|\!\sim}$-complete *if for every* $\mathrel{|\!\sim}$*-relation IR there is* $\Delta \in \mathcal{S}$ *with* $\mathrel{|\!\sim}_\Delta = IR$*. A normal form <NF> is* $\mathrel{|\!\sim}$-complete *if* $\boldsymbol{\Delta}$(*<NF>*) *is* $\mathrel{|\!\sim}$*-complete.*

All semantic approaches for conditionals cited above have in common that they are ignorant with respect to the syntax of formulas. Moreover, $\mathrel{|\!\sim}^p$, $\mathrel{|\!\sim}^z$, $\mathrel{|\!\sim}^c$, and $\mathrel{|\!\sim}^{\mathsf{w}}$are ignorant with respect to self-fulfilling conditionals; furthermore, they treat two conditionals having the same verification and the same falsification behaviour identically.

While the set of syntactically different conditionals and also the set of different belief bases over Σ is infinite because $\mathcal{L} = \mathcal{L}(\Sigma)$ is infinite, we can abstract from the syntactic variants of the underlying propositional language $\mathcal{L}$ and represent each formula $A \in \mathcal{L}$ uniquely by its set Ω_A of satisfying worlds. This allows us to simplify the representation of conditionals by using only *normal form conditionals*. In the following proposition, the two conditions $B \subsetneqq A$ and $B \neq \emptyset$ ensure both the falsifiability and the verifiability of a conditional $(B|A)$, thereby excluding any trivial conditional.

Proposition 1 ($NFC(\Sigma)$ (Beierle and Kutsch 2019b))**.** *For* $NFC(\Sigma) = \{(B|A) \mid A \subseteq \Omega_\Sigma,\ B \subsetneqq A,\ B \neq \emptyset\}$*, the set of* normal form conditionals *over* Σ*, the following holds: (i)* $NFC(\Sigma)$ *does not contain any trivial conditional. (ii) For every nontrivial conditional over* Σ *there is a conditionally equivalent conditional in* $NFC(\Sigma)$*. (iii) All conditionals in* $NFC(\Sigma)$ *are pairwise not conditionally equivalent.*

Example 1. *Representing each conditonal in* $R_0 = \{(\overline{a}|b), (b|a \vee b), (\overline{a} \vee b|a \vee \overline{b})\}$ *by its conditionally equivalent normal form conditional yields* $R_0' = \{(\{\overline{a}b\}|\{ab, \overline{a}b\}), (\{ab, \overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\}), (\{ab, \overline{a}\overline{b}\}|\{ab, a\overline{b}, \overline{a}\overline{b}\})\}$.

Using only $NFC(\Sigma)$-conditionals yields the CNF normal form, and for each Δ, there is a uniquely determined CNF.

Definition 4 (CNF, CNF(Δ) (Beierle and Haldimann 2022a))**.** *A belief base* Δ *over* Σ *is in* conditional normal form *(CNF) if* $\Delta \subseteq NFC(\Sigma)$*. For each consistent belief base* Δ *over* Σ*, its CNF representation is CNF*$(\Delta) = \{(\Omega_{AB}|\Omega_A) \mid (B|A) \in \Delta\} \cap NFC(\Sigma)$.

Normal form conditionals are sufficient to represent every $\mid\!\sim$-relation induced by an inductive inference operator $\mid\!\sim$ obeying (SF) and (CE).

Proposition 2 ((Beierle and Haldimann 2022a))**.** *CNF is* $\mid\!\sim$*-complete if* $\mid\!\sim$ *satisfies (SF) and (CE).*

Note that Proposition 2 covers all inductive inference operators discussed above, in particular, it covers $\mid\!\sim^p$, $\mid\!\sim^z$, $\mid\!\sim^c$, and $\mid\!\sim^{\mathsf{w}}$.

4 Observational Equivalence and Observational Normal Form

When comparing two belief bases Δ_1 and Δ_2, inferential equivalence $\equiv_{\mid\sim}$ with respect to an inductive inference operator (Definition 2) takes the full induced inference relations into account. However, in certain situations one may not be interested in the complete inference relations $\mid\!\sim_{\Delta_i}$ obtained from Δ_i via $\mid\!\sim$, but only whether Δ_1 and Δ_2 agree or disagree on a particular set of queries. For as much flexibility as possible, we consider conditionals as queries. Note that choosing conditionals here is more general than using propositional formulas because $A \in \mathcal{L}$ can be represented by $(A|\top)$.

Definition 5 (observational equivalence, $\equiv^Q_{\mid\sim}$)**.** *Let* $\mid\!\sim$ *be an inductive inference operator and let* $Q \subseteq NFC(\Sigma)$ *be a set of conditionals, called* queries*. Two belief bases* Δ, Δ' *over* Σ *are* observationally equivalent with respect to $\mid\!\sim$ and Q*, denoted by* $\Delta \equiv^Q_{\mid\sim} \Delta'$*, if, for all* $(B|A) \in Q$*, it holds that* $A \mid\!\sim_\Delta B$ *if and only if* $A \mid\!\sim_{\Delta'} B$.

For instance, one might be interested in the question whether Δ_1 and Δ_2 agree on all questions of the form whether a world ω is considered strictly more plausible than a world ω'. Questions of this form can be modelled by so-called *base conditionals*.

Definition 6. *The set of* base conditionals *over a signature* Σ *is given by:*

$$BC(\Sigma) = \{(B|A) \in NFC(\Sigma) \mid |A| = 2 \textit{ and } |B| = 1\}$$

Acceptance of the base conditional $(\omega|\omega \vee \omega')$ by a model of Δ corresponds to ω being considered strictly more plausible than ω'. For a ranking model κ, the following derivation makes this observation explicit:

$$\kappa \models (\omega|\omega \vee \omega') \tag{1}$$

$$\text{iff} \quad \kappa((\omega \vee \omega') \wedge \omega) < \kappa((\omega \vee \omega') \wedge \overline{\omega}) \tag{2}$$

$$\text{iff} \quad \kappa(\omega) < \kappa(\omega') \tag{3}$$

The following example illustrates observational equivalence with respect to base conditionals.

Example 2. *Let* $\Sigma_{ab} = \{a, b\}$*, and consider the following belief bases over* Σ_{ab}*:*

$$R_1 = \{(a\overline{b}|a\overline{b} \vee \overline{a}b)\} \tag{4}$$

$$R_3 = \{(a\overline{b}|a\overline{b} \vee \overline{a}b), (ab \vee \overline{a}\overline{b}|ab \vee a\overline{b} \vee \overline{a}\overline{b})\} \tag{5}$$

We have $a\overline{b} \vee \overline{a}b \mid\!\sim^p_{R_i} a\overline{b}$ *for* $i \in \{1, 3\}$*, and* $ab \vee a\overline{b} \vee \overline{a}\overline{b} \mid\!\sim^p_{R_3} ab \vee \overline{a}\overline{b}$*, but* $ab \vee a\overline{b} \vee \overline{a}\overline{b} \not\mid\!\sim^p_{R_1} ab \vee \overline{a}\overline{b}$*. While* $R_1 \not\equiv_{\mid\sim^p} R_3$*, for the set of base conditionals we have* $R_1 \equiv^{BC(\Sigma_{ab})}_{\mid\sim^p} R_3$.

Among all belief bases that are observationally equivalent with respect to an inductive inference operator and a set of queries we want to select a unique representative. For this, we employ an ordering on the set of belief bases over $NFC(\Sigma)$ as it is developed in (Beierle and Haldimann 2020a). This ordering uses signature renamings, where a function $\rho : \Sigma \to \Sigma$ is a *renaming* if ρ is a bijection. E.g., the function ρ_{ab} with $\rho_{ab}(a) = b$ and $\rho_{ab}(b) = a$ is a renaming for Σ_{ab}. As usual, ρ is extended canonically to worlds, formulas, conditionals, belief bases, and to sets thereof.

Definition 7 ($\simeq$)**.** *Let* X, X' *be two signatures, worlds, formulas, belief bases, sets, or relations over one of these items. We say that* X *and* X' *are* isomorphic with respect to signature renamings*, denoted by* $X \simeq X'$*, if there exists a renaming* ρ *such that* $\rho(X) = X'$.

For a set M, $m \in M$, and an equivalence relation $\equiv$ on M, the set of equivalence classes induced by $\equiv$ is denoted by $[M]_{/\equiv}$, and the unique equivalence class containing m is denoted by $[m]_\equiv$. E.g., $[\Omega_{\Sigma_{ab}}]_{/\simeq} = \{[ab]_\simeq, [a\overline{b}, \overline{a}b]_\simeq, [\overline{a}\overline{b}]_\simeq\}$ are the three equivalence classes of worlds over $\Sigma_{ab} = \{a, b\}$, and we have $[(ab|ab \vee a\overline{b})]_\simeq = [(ab|ab \vee \overline{a}b)]_\simeq$.

Based on the equivalence classes with respect to $\simeq$, the linear ordering $\prec$ on $NFC(\Sigma)$ is defined in (Beierle and Haldimann 2020a) for each ordered signature Σ. The $\prec$-minimal conditional in each equivalence class in $[NFC(\Sigma_{ab})]_{/\simeq}$ is the canonical representative of that class, called *canonical normal form conditional*. Extending $\prec$ yields an ordering on belief bases.

Definition 8 ($\Delta \preccurlyeq \Delta'$ (Beierle and Haldimann 2020a)). *The lexicographic extension of the ordering $\preccurlyeq$ on $NFC(\Sigma)$ to strings over $NFC(\Sigma)$ is denoted by $\preccurlyeq_{lex}$. For belief bases $\Delta = \{r_1, \ldots, r_n\}$ and $\Delta' = \{r'_1, \ldots, r'_{n'}\}$ over $NFC(\Sigma)$ with $r_i \prec r_{i+1}$ and $r'_j \prec r'_{j+1}$ the ordering $\preccurlyeq_{set}$ is given by: $\Delta \preccurlyeq_{set} \Delta'$ iff $n < n'$, or $n = n'$ and $r_1 \ldots r_n \preccurlyeq_{lex} r'_1 \ldots r'_{n'}$. Furthermore, $\Delta \preccurlyeq \Delta'$ stands for $\Delta \preccurlyeq_{set} \Delta'$.*

For instance, for the belief bases from Example 2 we have $R_1 \preccurlyeq R_3$. The ordering $\preccurlyeq$ on belief bases is used in the definition of observational normal form.

Definition 9 ($\text{ONF}^Q_{\mid\sim}$). *A belief base Δ is in* observational normal form with respect to $\mid\sim$ and a set of queries Q (in $\text{ONF}^Q_{\mid\sim}$)*, if Δ is in CNF and for every belief base Δ' in CNF with $\Delta \equiv^Q_{\mid\sim} \Delta'$ it holds that $\Delta \preccurlyeq \Delta'$.*

Example 3. *The belief base R_1 from Example 2 is in $ONF^{BC(\Sigma_{ab})}_{\mid\sim^p}$.*

Using the observation that $\preccurlyeq$ is a linear ordering on belief bases, yields the following proposition.

Proposition 3 ($\text{ONF}^Q_{\mid\sim}(\Delta)$). *Let $\mid\sim$ be an inductive inference operator and let Q be a set of queries. For every consistent Δ in CNF there is a uniquely determined belief base in $ONF^Q_{\mid\sim}$, denoted by $ONF^Q_{\mid\sim}(\Delta)$, with $\Delta \equiv^Q_{\mid\sim} ONF^Q_{\mid\sim}(\Delta)$.*

Example 4. *For the belief bases from Example 2 we have $R_1 = ONF^{BC(\Sigma_{ab})}_{\mid\sim^p}(R_3)$.*

In analogy to the completeness property of CNF (Prop. 2) we obtain that $\text{ONF}^Q_{\mid\sim}$ is $\mid\sim$-complete when considering $\mid\sim$-relations focussed on Q.

Proposition 4. *If $\mid\sim$ satisfies (SF) and (CE), then for every $\mid\sim$-relation IR there is a Δ in $ONF^Q_{\mid\sim}$ with $A \mid\sim_\Delta B$ iff $(A, B) \in IR \cap \{(A', B') \mid (B'|A') \in Q\}$.*

If we take all normal form conditionals as queries into account, observational equivalence and inferential equivalence coincide.

Proposition 5. *If $\mid\sim$ satisfies (SF) and (CE), then*

$$\equiv^{NFC(\Sigma)}_{\mid\sim} \;=\; \equiv_{\mid\sim} \; . \tag{6}$$

Using an algorithm for the systematic generation of belief bases (Beierle and Haldimann 2022b) and the reasoning system InfOCF (Kutsch and Beierle 2021; Kutsch 2019), leads us to the following observation.

Observation 1. *Systematically generating consistent belief bases over the two-element signature $\Sigma_{ab} = \{a, b\}$ and classifying them according to observational equivalence with respect to all normal form conditionals over Σ_{ab} yields to the follwing numbers:*

ind. inference operator $\mid\sim$		*belief bases in* $ONF^{NFC(\Sigma_{ab})}_{\mid\sim}$
p-entailment	$\mid\sim^p$	*485*
system Z	$\mid\sim^z$	*75*
c-inference	$\mid\sim^c$	*111*
system W	$\mid\sim^{\mathsf{w}}$	*75*

Another immediate consequence of observational equivalence is the following property.

Proposition 6. *Let $\mid\sim$ be an inductive inference operator and let Δ_1, Δ_2 be belief bases. If Q, Q' are sets of queries with $Q \subseteq Q'$ then $\Delta_1 \equiv^{Q'}_{\mid\sim} \Delta_2$ implies $\Delta_1 \equiv^Q_{\mid\sim} \Delta_2$.*

Note that the statement of Proposition 6 can also be compactly represented by the following equation:

$$Q \subseteq Q' \quad \text{implies} \quad \equiv^{Q'}_{\mid\sim} \subseteq \equiv^Q_{\mid\sim} \tag{7}$$

In the following section, we will focus on a specific set of queries; furthermore we will take renamings of belief bases into account.

5 Base Conditional Normal Form and Renamings on Belief Bases

As indicated in the previous section, the acceptance of base conditionals induces a partial order on worlds representing their relative plausibility, cf. Definition 6 and Equations (1)–(3). For the observational equivalence with respect to the set of all base conditionals we introduce a dedicated normal form.

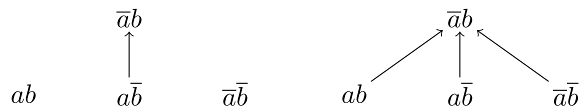
Definition 10 ($\text{BCNF}_{\mid\sim}$, $\text{BCNF}_{\mid\sim}(\Delta)$). *Let $\mid\sim$ be an inductive inference operator and let Q be a set of queries over a signature Σ. A belief base Δ over Σ is in* base conditional normal form with respect to $\mid\sim$ (in $\text{BCNF}_{\mid\sim}$)*, if Δ is in $ONF^{BC(\Sigma)}_{\mid\sim}$. The $BCNF_{\mid\sim}$ of Δ is $BCNF_{\mid\sim}(\Delta) = ONF^{BC(\Sigma)}_{\mid\sim}(\Delta)$.*

Observation 2. *Evaluating belief bases over Σ_{ab} with respect to base conditional normal form yields the following numbers:*

ind. inference operator $\mid\sim$		*belief bases in* $BCNF_{\mid\sim}$
p-entailment	$\mid\sim^p$	*219*
system Z	$\mid\sim^z$	*75*
c-inference	$\mid\sim^c$	*99*
system W	$\mid\sim^{\mathsf{w}}$	*75*

Thus, with p-entailment all 219 different strict partial orders that exist over the four Σ_{ab}-worlds can be obtained from a Σ_{ab}-belief base. All other inductive inference operators mentioned in Observation 2 are less skeptical than p-entailment; for each of these other operators, only a subset of the 219 strict partial orders can be modelled by a Σ_{ab}-belief base.

Example 5. *Consider the belief base R_1 from Example 2. The base conditional acceptance with respect to $\mid\sim^p$ and R_1 is illustrated in Figure 1(a). Each of $\mid\sim^\circ \in \{\mid\sim^z, \mid\sim^c, \mid\sim^{\mathsf{w}}\}$ entails additionally the two further strict order plausibility relations $ab \prec \overline{a}\overline{b}$ and $\overline{a}b \prec \overline{a}b$ (cf. Figure 1(b)). Note that there is no belief base such that either $\mid\sim^z$, $\mid\sim^c$, or $\mid\sim^{\mathsf{w}}$*



(a) The strict partial order on worlds corresponding to the base conditional acceptance of the inference relation induced by R_1 with $\mid\!\sim^p$.

(b) The strict partial order on worlds corresponding to the base conditional acceptance of the inference relation induced by R_1 with $\mid\!\sim^\circ \in \{\mid\!\sim^z, \mid\!\sim^c, \mid\!\sim^w\}$.

Figure 1: Strict partial orders on worlds induced by different inference relations from R_1 (cf. Example 5).

shows the acceptance behaviour with respect to base conditionals shown in Figure 1(a).

The number of systematically generated belief bases over a signature Σ can be reduced significantly by taking symmetries into account, for instance symmetries obtained by the permutation of propositional variables. Based on the renaming equivalence $\simeq$ (Definition 7) of belief bases and the linear ordering $\preccurlyeq$ (Definition 8), the following *renamnig normal form* (Beierle and Haldimann 2020a) has been proposed.

Definition 11 (ρNF, ρNF(Δ))**.** *A belief base Δ in CNF is in* renaming normal form *(ρNF) if for every Δ' with $\Delta \simeq \Delta'$ it holds that $\Delta \preccurlyeq \Delta'$. For every consistent Δ in CNF, the* renaming normal form ρNF(Δ) of Δ *is the uniquely determined belief base in ρNF such that $\Delta \simeq \rho NF(\Delta)$.*

Now it is interesting to combine BCNF$_{\mid\sim}$ and ρNF. However, the result of applying both BCNF$_{\mid\sim}$ and ρNF to a belief base depends on the order in which the normal forms are applied: in general, ρNF(BCNF$_{\mid\sim}$ (Δ)) and BCNF$_{\mid\sim}$ (ρNF(Δ)) may be different belief bases.

Example 6. *Consider the belief bases*

$$R_5 = \{(\overline{a}\overline{b}|ab \vee \overline{a}b \vee \overline{a}\overline{b}), (ab \vee a\overline{b}|ab \vee a\overline{b} \vee \overline{a}b)\} \quad (8)$$

$$R_6 = \{(\overline{a}\overline{b}|ab \vee a\overline{b} \vee \overline{a}\overline{b}), (ab \vee \overline{a}b|ab \vee a\overline{b} \vee \overline{a}b)\} \quad (9)$$

$$R_7 = \{(\overline{a}\overline{b}|ab \vee \overline{a}b \vee \overline{a}\overline{b})\} \quad (10)$$

$$R_8 = \{(\overline{a}\overline{b}|ab \vee a\overline{b} \vee \overline{a}\overline{b})\} \quad (11)$$

Note that R_7 is obtained from R_5 by removing the second conditional. We have $BCNF_{\mid\sim^p}(R_5) = R_7$ as the addition of the second conditional does not lead to the entailment of any other base conditionals. Regarding renamings, we have $\rho NF(R_7) = R_8$ as $R_8 = \rho_{ab}(R_7)$ and $R_8 \prec R_7$. Furthermore, we have $\rho NF(R_5) = R_5$ as $R_5 \prec \rho_{ab}(R_5) = R_6$. Hence, $\rho NF(BCNF_{\mid\sim^p}(R_5)) = R_8$ and $BCNF_{\mid\sim^p}(\rho NF(R_5)) = R_7$ but $R_8 \neq R_7$.

While Example 6 exhibits an undesirable phenomenon, the result of BCNF$_{\mid\sim}$ (ρNF(Δ)) may not even be in ρNF.

Example 7. *For the belief bases from Example 6, we have $BCNF_{\mid\sim^p}(\rho NF(R_5)) = R_7$; but R_7 is not in ρNF as R_8 is a renaming of R_7 with $R_8 \prec R_7$.*

6 Renamings on Observations

In the case of observational equivalence, instead of taking renamings of the involved belief bases into account, we now also consider renamings of the queries. Taking this perspective is useful, e.g., if the belief bases are not accessible to the observing agent.

Definition 12 ($\equiv^{[Q]}_{\mid\sim}$)**.** *Δ, Δ' are* observationally equivalent with respect to $\mid\!\sim$ and Q modulo renamings of the query variables*, denoted by $\Delta \equiv^{[Q]}_{\mid\sim} \Delta'$, if there is a renaming ρ such that for all $(B|A) \in Q$ it holds that $A \mid\!\sim_\Delta B$ if and only if $\rho(A) \mid\!\sim_{\Delta'} \rho(B)$.*

Example 8. *Consider the belief bases R_1, R_3 from Example 2 and*

$$R_2 = \{(\overline{a}b|a\overline{b} \vee \overline{a}b)\}. \quad (12)$$

For R_1 and R_2, we have $R_1 \equiv^{[BC(\Sigma_{ab})]}_{\mid\sim^p} R_2$. Additionally, $R_1 \simeq R_2$ because $\rho_{ab}(R_1) = R_2$.

In Example 8, there are two belief bases which are observationally equivalent with respect to $BC(\Sigma_{ab})$ modulo renamings of the query variables; additionally the belief bases are renamings of each other. The next proposition generalizes this correspondence for all renaming equivalent belief bases and arbitrary sets of queries.

Proposition 7 ($\simeq$ implies $\equiv^{[Q]}_{\mid\sim}$)**.** *Let $\mid\!\sim$ be an inductive inference operator and let Q be a set of queries. Then*

$$\Delta \simeq \Delta' \text{ implies } \Delta \equiv^{[Q]}_{\mid\sim} \Delta'. \quad (13)$$

However, the implication of Proposition 7 does not hold in the other direction.

Example 9. *Consider the belief bases R_1, R_2, R_3 from Examples 2 and 8 and*

$$R_4 = \{(\overline{a}b|a\overline{b} \vee \overline{a}b), (ab \vee \overline{a}\overline{b}|ab \vee a\overline{b} \vee \overline{a}\overline{b})\}. \quad (14)$$

For R_3 and R_4 we have observational equivalence with respect to the queries $BC(\Sigma_{ab})$ modulo renaming, i.e., $R_3 \equiv^{[BC(\Sigma_{ab})]}_{\mid\sim^p} R_4$. P-entailment with respect to the belief base R_3 accepts exactly the base conditional $r = (a\overline{b}|a\overline{b} \vee \overline{a}b)$ from $BC(\Sigma_{ab})$, and p-entailment with respect to R_4 accepts exactly the base conditional $r' = (\overline{a}b|a\overline{b} \vee \overline{a}b)$. The conditionals r and r' are equivalent under renaming because $\rho_{ab}(r) = r'$. However, R_3 and R_4 are not renaming equivalent because $R_3 \neq R_4$ and $\rho_{ab}(R_3) \neq R_4$.

We now introduce a normal form for belief bases based on the equivalence $\equiv^{[Q]}_{\mid\sim}$.

Definition 13 (πONF$^{[Q]}_{\mid\sim}$)**.** *Δ is in* observational normal form with respect to $\mid\!\sim$ and Q modulo renamings of the query variables (in πONF$^{[Q]}_{\mid\sim}$)*, if Δ is in CNF and for every Δ' in CNF with $\Delta \equiv^{[Q]}_{\mid\sim} \Delta'$ it holds that $\Delta \preccurlyeq \Delta'$.*

Example 10. *R_1 from Example 2 is in $\pi ONF^{[BC(\Sigma_{ab})]}_{\mid\sim^p}$ because there is no Δ with $\Delta \equiv^{[BC(\Sigma_{ab})]}_{\mid\sim^p} R_1$ and $\Delta \prec R_1$.*

For every inductive inference operator and any set of queries, Definition 13 induces a uniquely determined normal form.

Proposition 8 ($\pi\text{ONF}^{[Q]}_{|\!\sim}(\Delta)$)**.** *Let $|\!\sim$ be an inductive inference operator and let Q be a set of queries. For every consistent Δ in CNF there is a uniquely determined belief base in $\pi ONF^{[Q]}_{|\!\sim}$, denoted by $\pi ONF^{[Q]}_{|\!\sim}(\Delta)$, with $\Delta \equiv^{[Q]}_{|\!\sim} \pi ONF^{[Q]}_{|\!\sim}(\Delta)$.*

Example 11. *For R_2 from Example 8 the unique equivalent belief base in $\pi ONF^{[BC(\Sigma_{ab})]}_{|\!\sim^p}$ is R_1 from Example 2 (cf. Examples 8 and 10); i.e., $\pi ONF^{[BC(\Sigma_{ab})]}_{|\!\sim^p}(R_2) = R_1$.*

In our empirical evaluations, we determined observational equivalence moduo renamings over all normal form conditionals for Σ_{ab}-belief bases.

Observation 3. *Determining belief bases over Σ_{ab} in observational normal form with respect to all normal form conditionals modulo renaming of the query variables yields the following numbers:*

ind. inference operator $|\!\sim$		*belief bases in* $\pi ONF^{[NFC(\Sigma_{ab})]}_{|\!\sim}$
p-entailment	$|\!\sim^p$	*263*
system Z	$|\!\sim^z$	*44*
c-inference	$|\!\sim^c$	*63*
system W	$|\!\sim^{\mathsf{w}}$	*44*

Instead of combining observational equivalence with respect to the set $BC(\Sigma)$ of all base conditionals with renamings of *belief bases* (cf. Examples 6 and 7), the following normal form combines observational equivalence with respect to $BC(\Sigma)$ with renamings of the *query variables*.

Definition 14 ($\pi\text{BCNF}_{|\!\sim}$, $\pi\text{BCNF}_{|\!\sim}(\Delta)$)**.** *A belief base Δ is in* base conditional normal form with respect to $|\!\sim$ modulo renamings of the query variables (in $\pi\text{BCNF}_{|\!\sim}$)*, if Δ is in $\pi ONF^{[BC(\Sigma)]}_{|\!\sim}$. The $\pi BCNF_{|\!\sim}$ of a belief base Δ is the uniquely determined $\pi BCNF_{|\!\sim}(\Delta) = \pi ONF^{[BC(\Sigma)]}_{|\!\sim}(\Delta)$.*

Example 12. *R_1 from Example 2 is in $\pi BCNF_{|\!\sim^p}$ (cf. Example 10).*

Observation 4. *Evaluating belief bases over Σ_{ab} with respect to base conditional normal form modulo renamings of the query variables yields the following numbers:*

ind. inference operator $|\!\sim$		*belief bases in* $\pi BCNF_{|\!\sim}$
p-entailment	$|\!\sim^p$	*119*
system Z	$|\!\sim^z$	*44*
c-inference	$|\!\sim^c$	*56*
system W	$|\!\sim^{\mathsf{w}}$	*44*

In Example 6 and 7 we demonstrated phenomena that may occur when combining the normal forms $\pi\text{BCNF}_{|\!\sim}$ and ρNF arbitrarily. The next proposition relies on the correspondence expressed in (13) in Proposition 7 and clarifies the general correspondence among ρNF, $\text{ONF}^{BC(\Sigma)}_{|\!\sim}$, and $\pi\text{ONF}^{[BC(\Sigma)]}_{|\!\sim}$.

Proposition 9. *Let $|\!\sim$ be an inductive inference operator and Q be a set of queries. Then for any belief base Δ we have:*

$$\rho NF(ONF^{Q}_{|\!\sim}(\Delta)) = \pi ONF^{[Q]}_{|\!\sim}(\Delta) \tag{15}$$

Thus, given any belief base Δ, first determining $\text{ONF}^{Q}_{|\!\sim}(\Delta)$ and then transforming the result into ρNF yields $\pi\text{ONF}^{[Q]}_{|\!\sim}(\Delta)$. Because $\text{BCNF}_{|\!\sim}$ and $\pi\text{BCNF}_{|\!\sim}$ are special cases of $\text{ONF}^{Q}_{|\!\sim}$ and $\pi\text{ONF}^{[Q]}_{|\!\sim}$, respectively, an immediate consequence of Proposition 9 is:

Proposition 10. *Let $|\!\sim$ be an inductive inference operator. Then for any belief base Δ we have:*

$$\rho NF(BCNF_{|\!\sim}(\Delta)) = \pi BCNF_{|\!\sim}(\Delta) \tag{16}$$

Other immediate consequences of Proposition 9 are:

$$\rho\text{NF}(\pi\text{ONF}^{[Q]}_{|\!\sim}(\Delta)) = \pi\text{ONF}^{[Q]}_{|\!\sim}(\Delta) \tag{17}$$

$$\rho\text{NF}(\pi\text{BCNF}_{|\!\sim}(\Delta)) = \pi\text{BCNF}_{|\!\sim}(\Delta) \tag{18}$$

Thus, every $\pi\text{ONF}^{[Q]}_{|\!\sim}$ and hence every $\pi\text{BCNF}_{|\!\sim}$ is also in renaming normal form.

7 Conclusion and Future Work

In this paper, we introduced observational equivalence $\equiv^{Q}_{|\!\sim^p}$ with respect to an inductive inference operator $|\!\sim$ and a set of queries Q. Using this concept, we introduced the observational normal form $\text{ONF}^{Q}_{|\!\sim}$. By additionally considering renamings of the queries, we obtained observational equivalence modulo renaming $\equiv^{[Q]}_{|\!\sim^p}$ and the corresponding observational normal form modulo renamings $\pi\text{ONF}^{[Q]}_{|\!\sim}$. For obtaining the plausibility orderings on worlds that can be realized by applying an inductive inference operator to a belief base, we introduced the base conditional normal form $\text{BCNF}_{|\!\sim}$ and the base conditional normal form modulo renamings $\pi\text{BCNF}_{|\!\sim}$ as special cases of the $\text{ONF}^{Q}_{|\!\sim}$ and the $\pi\text{ONF}^{[Q]}_{|\!\sim}$, respectively.

We elaborated the general interrelationships among these normal forms. Our empirical evaluations yield the number of belief bases in each normal form over the signature Σ_{ab} for each of the four inductive inference operators p-entailment, system Z, c-inference, and system W. The evaluations show, for example, that p-entailment can realize all 219 strict partial orders on the four Σ_{ab} worlds, while with each of the other three considered inductive inference operators only a subset of these possible plausibility orderings on worlds can be achieved.

Our future work includes the generation and empirical evaluation of belief bases in these normal forms for larger signatures as well as the coverage of other inductive inference operators.

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References

Adams, E. 1965. The Logic of Conditionals. *Inquiry* 8(1-4):166–197.

Beierle, C., and Haldimann, J. 2020a. Normal forms of conditional knowledge bases respecting system P-entailments. In Herzig, A., and Kontinen, J., eds., *FoIKS 2020*, volume 12012 of *LNCS*, 22–41. Springer.

Beierle, C., and Haldimann, J. 2020b. Transforming conditional knowledge bases into renaming normal form. In Barták, R., and Bell, E., eds., *Proceedings of the 33rd International Florida Artificial Intelligence Research Society Conference, FLAIRS 2020*, 563–568. AAAI Press.

Beierle, C., and Haldimann, J. 2022a. Normal forms of conditional belief bases respecting inductive inference. In Keshtkar, F., and Franklin, M., eds., *Proceedings of the Thirty-Fifth International Florida Artificial Intelligence Research Society Conference (FLAIRS), Hutchinson Island, Florida, USA, May 15-18, 2022*.

Beierle, C., and Haldimann, J. 2022b. Normal forms of conditional knowledge bases respecting system P-entailments and signature renamings. *Ann. Math. Artif. Intell.* 90(2):149–179.

Beierle, C., and Kutsch, S. 2019a. On the antecedent normal form of conditional knowledge bases. In *ECSQARU 2019*, volume 11762 of *LNAI*, 175–186. Springer.

Beierle, C., and Kutsch, S. 2019b. Systematic generation of conditional knowledge bases up to renaming and equivalence. In *JELIA 2019*, volume 11468 of *LNAI*, 279–286. Springer.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2018. Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem. *Ann. Math. Artif. Intell.* 83(3-4):247–275.

Beierle, C.; Kern-Isberner, G.; Sauerwald, K.; Bock, T.; and Ragni, M. 2019. Towards a general framework for kinds of forgetting in common-sense belief management. *KI – Künstliche Intelligenz* 33(1):57–68.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2021. Properties and interrelationships of skeptical, weakly skeptical, and credulous inference induced by classes of minimal models. *Artificial Intelligence* 297:103489.

Beierle, C.; Haldimann, J.; and Kutsch, S. 2021. A complete map of conditional knowledge bases in different normal forms and their induced system P inference relations over small signatures. In *Proceedings of the 34th International Florida Artificial Intelligence Research Society Conference, FLAIRS 2021*.

Beierle, C. 2019. Inferential equivalence, normal forms, and isomorphisms of knowledge bases in institutions of conditional logics. In Hung, C., and Papadopoulos, G. A., eds., *The 34th ACM/SIGAPP Symposium on Applied Computing (SAC '19), April 8–12, 2019, Limassol, Cyprus*, 1131–1138. New York, NY, USA: ACM.

Benferhat, S.; Dubois, D.; and Prade, H. 1999. Possibilistic and standard probabilistic semantics of conditional knowledge bases. *J. of Logic and Computation* 9(6):873–895.

Dubois, D., and Prade, H. 1994. Conditional objects as nonmonotonic consequence relationships. *Special Issue on Conditional Event Algebra, IEEE Transactions on Systems, Man and Cybernetics* 24(12):1724–1740.

Eiter, T., and Kern-Isberner, G. 2019. A brief survey on forgetting from a knowledge representation and reasoning perspective. *KI – Künstliche Intelligenz* 33(1):9–33.

Goldszmidt, M., and Pearl, J. 1996. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artif. Intell.* 84:57–112.

Kern-Isberner, G.; Beierle, C.; and Brewka, G. 2020. Syntax splitting = relevance + independence: New postulates for nonmonotonic reasoning from conditional belief bases. In *KR-2020*, 560–571.

Kern-Isberner, G. 2001. *Conditionals in nonmonotonic reasoning and belief revision*, volume 2087 of *LNAI*. Springer.

Kern-Isberner, G. 2004. A thorough axiomatization of a principle of conditional preservation in belief revision. *Ann. Math. Artif. Intell.* 40(1-2):127–164.

Komo, C., and Beierle, C. 2020. Nonmonotonic inferences with qualitative conditionals based on preferred structures on worlds. In Schmid, U.; Klügl, F.; and Wolter, D., eds., *KI 2020: Advances in Artificial Intelligence - 43rd German Conference on AI, Bamberg, Germany, September 21-25, 2020, Proceedings*, volume 12325 of *LNCS*, 102–115. Springer.

Komo, C., and Beierle, C. 2022. Nonmonotonic reasoning from conditional knowledge bases with system W. *Ann. Math. Artif. Intell.* 90(1):107–144.

Kraus, S.; Lehmann, D. J.; and Magidor, M. 1990. Nonmonotonic Reasoning, Preferential Models and Cumulative Logics. *Artificial Intelligence* 44(1-2):167–207.

Kutsch, S., and Beierle, C. 2021. InfOCF-Web: An online tool for nonmonotonic reasoning with conditionals and ranking functions. In Zhou, Z., ed., *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021*, 4996–4999. ijcai.org.

Kutsch, S. 2019. InfOCF-Lib: A Java library for OCF-based conditional inference. In *Proc. DKB/KIK-2019*, volume 2445 of *CEUR Workshop Proceedings*, 47–58. CEUR-WS.org.

Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artif. Intell.* 55:1–60.

Lewis, D. 1973. *Counterfactuals*. Cambridge, Mass.: Harvard University Press.

Reiter, R. 1980. A logic for default reasoning. *Artificial Intelligence* 13:81–132.

Spohn, W. 1988. Ordinal conditional functions: a dynamic theory of epistemic states. In Harper, W., and Skyrms, B., eds., *Causation in Decision, Belief Change, and Statistics, II*. Kluwer Academic Publishers. 105–134.

Spohn, W. 2012. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford, UK: Oxford University Press.