Conditionals, Infeasible Worlds, and Reasoning with System W

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Abstract

The recently introduced notion of an inductive inference operator captures the process of completing a given conditional belief base to an inference relation. System W is such an inductive inference operator exhibiting some notable properties like extending rational closure and satisfying syntax splitting for inference from conditional belief bases. However, the definition of system W and the shown results regarding its properties only take belief bases into account that satisfy a strong notion of consistency where no worlds may be completely infeasible. In this paper, we lift this limitation and extend the definition of system W to also cover belief bases that force some worlds to be infeasible. We establish the position of the extended system W within a map of other inductive inference operators being able to deal with the presence of infeasible worlds, including system Z and multipreference closure. For placing lexicographic inference in this map, we show that the definition of lexicographic inference must be slightly modified so that it is an inductive inference operator satisfying direct inference even when there are worlds that are infeasible. Furthermore, we show that, like its unextended version, the extended system W enjoys other desirable properties such as still fully complying with syntax splitting.

1 Introduction

Completing a conditional belief base to an inference relation is a form of inductive inference; this process can be formally captured by inductive inference operators as introduced in (Kern-Isberner, Beierle, and Brewka 2020). One such inductive inference operator is system W (Komo and Beierle 2022). System W is an extension of both system Z (Pearl 1990) (or equivalently rational closure (Lehmann 1989)) and skeptical c-inference (Beierle et al. 2018; 2021), and it was shown to be extended by lexicographic inference (Lehmann 1995) in (Haldimann and Beierle 2022b). Additionally, system W satisfies syntax splitting for inductive inference operators from (Kern-Isberner, Beierle, and Brewka 2020), see (Haldimann and Beierle 2022b).

However, the definition of system W as well as the mentioned results only consider belief bases that satisfy a strong notion of consistency: the belief base may not require a world to be completely infeasible (the notion called strong consistency in this paper coincides with the notion of consistency in (Goldszmidt and Pearl 1996)). For instance, any belief base containing both the conditionals $(B|A)$ and $(\neg B|A)$ does not satisfy this notion of consistency as any world satisfying A would need to be infeasible. This is an unfortunate limitation because other approaches to inference, such as rational closure, lexicographic inference, or multipreference closure also cover belief bases that do not satisfy this strong notion of consistence. Especially in practical applications, belief bases may not always be carefully designed to be strongly consistent, but are possibly assembled from different, disagreeing sources or based on imperfect observations.

In this paper we generalize the definition of system W to system W$^+$ that also covers belief bases that force some worlds to be infeasible. Then we re-establish results shown for system W for the extended notion of system W. We show that system W$^+$ extends rational closure and coincides with multipreference closure (MP-closure) (Giordano and Gliozzi 2021) that has been proposed for reasoning in description logics with exceptions. We note that the definition of lexicographic inference has to be slightly modified for it to be an inductive inference operation; and we then show that the extended system W$^+$ is captured by lexicographic inference. Furthermore, we extend the syntax spliting postulates from (Kern-Isberner, Beierle, and Brewka 2020) to also cover weakly consistent belief bases and show that system W$^+$ fully complies with syntax splitting.

In summary the main contributions of this paper are:

- Contrasting juxaposition of the notions of weak and strong consistency for belief bases and systematic coverage of both notions.
- Extension of system W for weakly consistent belief bases.
- Placement of system W$^+$ in a map of inductive inference operators.
- Syntax splitting postulates for inference also from weakly consistent belief bases.
- Showing that system W$^+$ satisfies syntax splitting.

After recalling conditional logic in Sec. 2 and inductive inference in Sec. 3, we develop the extended system W$^+$ in Sec. 4. We show the connections of system W$^+$ to other inductive inference operators in Sec. 5, and show that extended system W satisfies syntax splitting in Sec. 6. Sec. 7 concludes and points out future work.

2 Conditional Logic

A *(propositional) signature* is a finite set Σ of propositional variables. Assuming an underlying signature Σ, we denote the resulting propositional language by $\mathcal{L}_\Sigma$. Usually, we denote elements of signatures with lowercase letters $a, b, c, \ldots$ and formulas with uppercase letters $A, B, C, \ldots$. We may denote a conjunction $A \wedge B$ by AB and a negation $\neg A$ by $\overline{A}$ for brevity of notation. The set of interpretations over the underlying signature is denoted as Ω_Σ. Interpretations are also called *worlds* and Ω_Σ the *universe*. An interpretation $\omega \in \Omega_\Sigma$ is a *model* of a formula $A \in \mathcal{L}$ if A holds in ω, denoted as $\omega \models A$. The set of models of a formula (over a signature Σ) is denoted as $\mathit{Mod}_\Sigma(A) = \{\omega \in \Omega_\Sigma \mid \omega \models A\}$ or short as Ω_A. The Σ in Ω_Σ, $\mathcal{L}_\Sigma$ and $\mathit{Mod}_\Sigma(A)$ can be omitted if the signature is clear from the context or if the underlying signature is not relevant. A formula A *entails* a formula B if $\Omega_A \subseteq \Omega_B$. By slight abuse of notation we sometimes interpret worlds as the corresponding complete conjunction of all elements in the signature in either positive or negated form.

A *conditional* $(B|A)$ connects two formulas A, B and represents the rule "If A then usually B", where A is called the *antecedent* and B the *consequent* of the conditonal. The conditional language is denoted as $(\mathcal{L}|\mathcal{L})_\Sigma = \{(B|A) \mid A, B \in \mathcal{L}_\Sigma\}$. A finite set of conditionals is called a *belief base*. We use a three-valued semantics of conditionals in this paper (de Finetti 1937). For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. Popular models for belief bases are ranking functions (also called ordinal conditional functions, OCF) (Spohn 1988) and total preorders (TPO) on Ω_Σ (Darwiche and Pearl 1997). An OCF $\kappa : \Omega_\Sigma \to \mathbb{N}_0 \cup \{\infty\}$ maps worlds to a *rank* such that at least one world has rank 0, i.e., $\kappa^{-1}(0) \neq \emptyset$. The intuition is that worlds with lower ranks are more plausible than worlds with higher ranks; worlds with rank ∞ are considered infeasible. OCFs are lifted to formulas by mapping a formula A to the smallest rank of a model of A, or to ∞ if A has no models. An OCF κ is a model of a conditional $(B|A)$, denoted as $\kappa \models (B|A)$, if $\kappa(A) = \infty$ or if $\kappa(AB) < \kappa(A\overline{B})$; κ is a model of a belief base Δ, denoted as $\kappa \models \Delta$, if it is a model of every conditional in Δ.

Note that there are different definitions of consistency in the literature. To distinguish two different notions of consistency that both occur in this paper we call one notion of consistency *strong consistency* and the other notion *weak consistency*.

Definition 1. *A belief base Δ is called* strongly consistent *if there exists at least one ranking function κ with $\kappa \models \Delta$ and $\kappa^{-1}(\infty) = \emptyset$. A belief base Δ is* weakly consistent *if there is at least one ranking function κ with $\kappa \models \Delta$.*

Thus, Δ is strongly consistent if there is at least one ranking function modelling Δ that considers all worlds feasible. This notion of consistency is used in many approaches, e.g., (Goldszmidt and Pearl 1996). The notion of weak consistency is equivalent to the more relaxed notion of consistency that is used in, e.g., (Giordano et al. 2015; Casini, Meyer, and Varzinczak 2019). Trivially, strong consistency implies weak consistency.

3 Inductive Inference

The conditional beliefs of an agent are formally captured by a binary relation $\mid\!\sim$ on propositional formulas with $A \mid\!\sim B$ representing that A (defeasibly) entails B; this relation is called *inference* or *entailment relation*.

Different sets of properties for inference relations have been suggested in literature; often the set of postulates called *system P* is considered as minimal requirement for inference relations. Inference relations satisfying system P are called *preferential inference relations* (Adams 1975; Kraus, Lehmann, and Magidor 1990).

Besides ranking functions, preferential models are another class of models for conditionals that are useful in the context of preferential inference relations.

Definition 2 (preferential model (Kraus, Lehmann, and Magidor 1990))**.** *A* preferential model *is a triple $\mathcal{M} = \langle S, l, \prec\rangle$ consisting of a set S of* states*, a function $l : S \to \Omega$ mapping states to interpretations, and a strict partial order $\prec$ on S. For $A \in \mathcal{L}$ and $s \in S$ we denote $l(s) \models A$ by $s \models\!\!\models A$; and we define $[\![A]\!]_\mathcal{M} = \{s \in S \mid s \models\!\!\models A\}$.*

Note that the definition of preferential models in (Kraus, Lehmann, and Magidor 1990) includes a *smothness condition*. As this condition is automatically satisfied for finite sets of interpretations as considered in this paper, it is omitted in Definition 2. A preferential model $\mathcal{M} = \langle S, l, \prec\rangle$ induces an inference relation $\mid\!\sim_\mathcal{M}$ by

$$A \mid\!\sim_\mathcal{M} B \quad \text{iff} \quad \min([\![A]\!]_\mathcal{M}, \prec) \subseteq [\![B]\!]_\mathcal{M}. \tag{1}$$

One remarkable result from (Kraus, Lehmann, and Magidor 1990) states that preferential models characterize preferential entailment relations: Every inference relation $\mid\!\sim_\mathcal{M}$ induced by a preferential model $\mathcal{M}$ is preferential, and for every preferential inference relation $\mid\!\sim$ there is a preferential model $\mathcal{M}$ with $\mid\!\sim_\mathcal{M} = \mid\!\sim$.

Inductive inference is the process of completing a given belief base to an inference relation. To formally capture this we use the concept of inductive inference operators.

Definition 3 (inductive inference operator (Kern-Isberner, Beierle, and Brewka 2020))**.** *An* inductive inference operator *is a mapping $C : \Delta \mapsto \mid\!\sim_\Delta$ that maps each belief base to an inference relation such that direct inference (DI) and trivial vacuity (TV) are fulfilled, i.e.,*

(DI) *if $(B|A) \in \Delta$ then $A \mid\!\sim_\Delta B$, and*

(TV) *if $\Delta = \emptyset$ and $A \mid\!\sim_\Delta B$ then $A \models B$.*

An inductive inference operator C is a *preferential inductive inference operator* if every inference relation $\mid\!\sim_\Delta$ in the image of C satisfies system P. Using preferential models we can define p-entailment (Kraus, Lehmann, and Magidor 1990) as a preferential inductive inference operator.

Definition 4 (p-entailment)**.** *Let Δ be a belief base and $A, B \in \mathcal{L}$. A* p-entails *B w.r.t. Δ, denoted $A \mid\!\sim^p_\Delta B$, if $A \mid\!\sim_\mathcal{M} B$ for any preferential model $\mathcal{M}$ of Δ. P-entailment is the inductive inference operator mapping each Δ to $\mid\!\sim^p_\Delta$.*

Weak consistency can be characterized by p-entailment.

Proposition 1. *Δ is weakly consistent iff $\top \not\mid\!\sim^p_\Delta \bot$.*

A preferential model $\mathcal{M}$ is called *canonical* (for a belief base Δ) if for every $\omega \in \Omega_\Sigma$ with $\omega \not\mathrel{|\!\sim}^p_\Delta \bot$ it holds that $[\![\omega]\!]_\mathcal{M} \neq \emptyset$.

The *inference relation* $\mathrel{|\!\sim}_\kappa$ *induced by a ranking function* κ is defined by

$$A \mathrel{|\!\sim}_\kappa B \quad \text{iff} \quad \kappa(A) = \infty \text{ or } \kappa(AB) < \kappa(A\overline{B}). \tag{2}$$

Note that the condition $\kappa(A) = \infty$ in (2) ensures that system P's axiom (Reflexivity) is satisfied for $A \equiv \bot$. System Z is an inductive inference operator that is defined based on the Z-partition (Pearl 1990). Here we use an extended version of system Z that also covers belief bases that are weakly consistent and that was shown to be equivalent to *rational closure* (Lehmann 1989) in (Goldszmidt and Pearl 1990).

Definition 5 ((extended) Z-partition)**.** *A conditional* $(B|A)$ *is* tolerated *by* $\Delta = \{(B_i|A_i) \mid i = 1, \dots, n\}$ *if there exists a world* $\omega \in \Omega$ *such that* ω *verifies* $(B|A)$ *and* ω *does not falsify any conditional in* Δ*, i.e.,* $\omega \models AB$ *and* $\omega \models \bigwedge_{i=1}^n (\overline{A_i} \vee B_i)$*.*

The (extended) Z-partition $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$ *of a belief base* Δ *is the ordered partition of* Δ *that is constructed by letting* Δ^i *be the inclusion maximal subset of* $\bigcup_{j=i}^n \Delta^j$ *that is tolerated by* $\bigcup_{j=i}^n \Delta^j$ *until* $\Delta^{k+1} = \emptyset$*. The set* Δ^∞ *is the remaining set of conditionals that contains no conditional which is tolerated by* Δ^∞*.*

It is well-known that the construction of $EZP(\Delta)$ is successful with $\Delta^\infty = \emptyset$ iff Δ is strongly consistent, and because the Δ^i are chosen inclusion-maximal, the Z-partition is unique (Pearl 1990). Also, it holds that $EZP(\Delta)$ has a $\Delta^0 \neq \emptyset$ iff Δ is weakly consistent.

Definition 6 ((extended) system Z)**.** *Let* Δ *be a belief base with* $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$*. If* Δ *is not weakly consistent, then let* $A \mathrel{|\!\sim}^z_\Delta B$ *for any* $A, B \in \mathcal{L}$*.*

Otherwise, the (extended) Z-ranking function κ^z_Δ *is defined as follows: For* $\omega \in \Omega$*, if one of the conditionals in* Δ^∞ *is applicable to* ω *define* $\kappa^z_\Delta(\omega) = \infty$*. If not, let* Δ^j *be the last partition in* $EZP(\Delta)$ *that contains a conditional falsified by* ω*. Then let* $\kappa^z_\Delta(\omega) = j + 1$*. If* ω *does not falsify any conditional in* Δ*, then let* $\kappa^z_\Delta(\omega) = 0$*.* (Extended) system Z *maps* Δ *to the inference relation* $\mathrel{|\!\sim}^z_\Delta$ *induced by* κ^z_Δ*.*

For strongly consistent belief bases the extended system Z coincides with system Z as defined in (Pearl 1990; Goldszmidt and Pearl 1996). Note that for any belief base Δ the induced κ^z_Δ is a model of Δ.

Lemma 1. *For a weakly consistent belief base* Δ *and a formula* A *we have* $\kappa^z_\Delta(A) = \infty$ *iff* $A \mathrel{|\!\sim}^p_\Delta \bot$*.*

Lemma 2. *Let* Δ *be a belief base with* $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$*. A world* $\omega \in \Omega$ *falsifies a conditional in* Δ^∞ *iff it is applicable for a conditional in* Δ^∞*.*

4 Extending System W

The definition of system W in (Komo and Beierle 2020; 2022) utilizes a strict partial order (SPO) called *preferred structure on worlds* on the set of all worlds Ω. To accommodate infeasible worlds, here we allow the preferred structure on worlds to order only a subset of Ω. Thus, we will use an extended notion of *SPO model* that is a strict partial order over a set of *feasible* worlds $\Omega^{feas} \subseteq \Omega_\Sigma$. For an SPO model $\prec$ over a set Ω^{feas}, a world ω is feasible w.r.t. $\prec$ if $\omega \in \Omega^{feas}$, and a formula F is feasible w.r.t. $\prec$ if at least one model of F is feasible. The following definition of the *preferred structures on worlds* is adapted from (Komo and Beierle 2022) to use this extended notion of SPO model instead of complete SPOs over Ω.

Definition 7 (ξ^j, ξ, preferred structure $<^{\mathsf{w}+}_\Delta$ on worlds)**.** *Let* Δ *be a belief base with the Z-partition* $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$*. For* $j = 0, \dots, k, \infty$ *the functions* ξ^j *and* ξ *are the* functions mapping worlds to the set of falsified conditionals *in* Δ^j *or* Δ*, respectively, given by*

$$\xi^j(\omega) := \{(B_i|A_i) \in \Delta^j \mid \omega \models A_i\overline{B_i}\}$$
$$\xi(\omega) := \{(B_i|A_i) \in \Delta \mid \omega \models A_i\overline{B_i}\}.$$

Let $\Omega^{feas} = \Omega \backslash \{\omega \mid \xi^\infty(\omega) \neq \emptyset\}$*. The* preferred structure on worlds *is the relation* $<^{\mathsf{w}+}_\Delta \subseteq \Omega^{feas} \times \Omega^{feas}$ *defined by*

$$\omega <^{\mathsf{w}+}_\Delta \omega' \text{ iff there exists an } m \in \{0, \dots, k\} \text{ such that}$$
$$\xi^i(\omega) = \xi^i(\omega') \quad \forall i \in \{m+1, \dots, k\} \text{ and}$$
$$\xi^m(\omega) \subsetneqq \xi^m(\omega').$$

Every belief base Δ induces a preferred structure on worlds $<^{\mathsf{w}+}_\Delta$. We have that $\omega <^{\mathsf{w}+}_\Delta \omega'$ if and only if ω falsifies strictly fewer conditionals than ω' in the partition with the highest index m where the conditionals falsified by ω and ω' differ. The relation $<^{\mathsf{w}+}_\Delta$ is an SPO model over Ω^{feas}. A world ω is feasible for $<^{\mathsf{w}+}_\Delta$ iff $\xi^\infty(\omega) = \emptyset$.

Definition 8 (system W$^+$, $\mathrel{|\!\sim}^{\mathsf{w}+}_\Delta$ (adapted from (Komo and Beierle 2022)))**.** *Let* Δ *be a belief base and* $A, B \in \mathcal{L}$*. Then* B *is a* system-W$^+$ inference *from* A*, denoted* $A \mathrel{|\!\sim}^{\mathsf{w}+}_\Delta B$*, if for every feasible* $\omega' \in \Omega_{A\overline{B}}$ *there is a feasible* $\omega \in \Omega_{AB}$ *such that* $\omega <^{\mathsf{w}+}_\Delta \omega'$*.*

For strongly consistent belief bases this definition of system W$^+$ coincides with the definition of system W given in (Komo and Beierle 2022).

Proposition 2. *Let* Δ *be a strongly consistent belief base and* $A, B \in \mathcal{L}$*. We have* $A \mathrel{|\!\sim}^{\mathsf{w}}_\Delta B$ *iff* $A \mathrel{|\!\sim}^{\mathsf{w}+}_\Delta B$*.*

Just as for system W we can represent every inference relation induced by system W$^+$ with a preferential model (see (Haldimann and Beierle 2022a)). For a belief base Δ the *system-W$^+$ preferential model* is $\mathcal{M}^{\mathsf{w}}(\Delta) = \langle \Omega^{feas}, \mathrm{id}, <^{\mathsf{w}+}_\Delta \rangle$. The inference relation induced by $\mathcal{M}^{\mathsf{w}}(\Delta)$ coincides with system-W$^+$ inference from Δ:

$$A \mathrel{|\!\sim}_{\mathcal{M}^{\mathsf{w}}(\Delta)} B \quad \text{iff} \quad A \mathrel{|\!\sim}^{\mathsf{w}+}_\Delta B.$$

This implies that system W$^+$ is a preferential inductive inference operator. Moreover, $\mathcal{M}^{\mathsf{w}}(\Delta)$ is also a canonical model.

Proposition 3. *The system-W$^+$ preferential model* $\mathcal{M}^{\mathsf{w}}(\Delta)$ *is a canonical model of* Δ*.*

In (Haldimann and Beierle 2022b) it was shown that system W satisfies weak rational monotony. This also holds for system W$^+$.

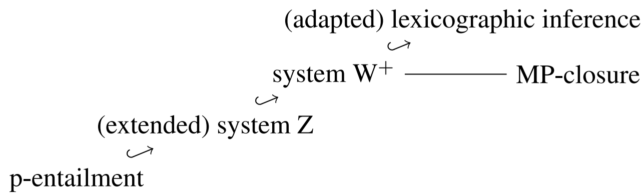

Figure 1: Overview over relationships among inductive inference operators. An arrow $A \hookrightarrow B$ indicates that inference operator A is captured and strictly extended by B.

Proposition 4. *System W$^+$ fulfils weak rational monotony (WRM), i.e., for any $A, B \in \mathcal{L}$ it holds that $\top \mathrel{|\!\sim} B$ and $\top \mathrel{|\!\not\sim} \overline{A}$ imply $A \mathrel{|\!\sim} B$.*

System W and therefore also system W$^+$ do not satisfy rational monotony (RM) or semi monotony (SM) (Haldimann and Beierle 2022b). Cautious monotony (CM) is satisfied as system W$^+$ is a preferential inductive inference operator.

5 System W$^+$ in Relation to Other Inductive Inference Operators

Previous papers connected system W (as defined in (Komo and Beierle 2022)) to other inductive inference operators: System W captures system Z (Komo and Beierle 2022), is captured by lexicographic inference (Haldimann and Beierle 2022b), and coincides with MP-closure (Haldimann and Beierle 2022a). In this section, we show that the system W$^+$ is also correspondingly connected to (adapted versions of) these inference relations. Figure 1 gives an overview over the relations among the inference operators.

5.1 Relation to (Extended) System Z

Just as the original version, system W$^+$ also captures (extended) system Z and thus rational closure.

Proposition 5. *System W$^+$ captures (extended) system Z, i.e., for a belief base Δ and $A, B \in \mathcal{L}$ it holds that $A \mathrel{|\!\sim}^{z}_{\Delta} B$ implies $A \mathrel{|\!\sim}^{\mathsf{w}+}_{\Delta} B$.*

Proof. Observing the definition of κ^z_Δ and $<^{\mathsf{w}+}_\Delta$ we see that for any $\omega, \omega' \in \Omega$

- $\kappa^z_\Delta(\omega) = \infty$ implies that ω is not feasible for $<^{\mathsf{w}+}_\Delta$ and
- $\kappa^z_\Delta(\omega) < \kappa^z_\Delta(\omega')$ implies $\omega <^{\mathsf{w}+}_\Delta \omega'$.

If $A \mathrel{|\!\sim}^{z}_{\Delta} B$ then either $\kappa^z_\Delta(A) = \infty$ or $\kappa^z_\Delta(AB) < \kappa^z_\Delta(A\overline{B})$. In the first case, A has no feasible model in $<^{\mathsf{w}+}_\Delta$, and therefore $A\overline{B}$ has no feasible model in $<^{\mathsf{w}+}_\Delta$. Trivially, $A \mathrel{|\!\sim}^{\mathsf{w}+}_{\Delta} B$. In the second case, there is an $\omega \in \Omega_{AB}$ with a smaller z-rank than any $\omega' \in \Omega_{A\overline{B}}$. This implies that for any $\omega' \in \Omega_{A\overline{B}}$ we have $\omega <^{\mathsf{w}+}_\Delta \omega'$. Therefore, $A \mathrel{|\!\sim}^{\mathsf{w}+}_{\Delta} B$. □

5.2 Relation to Lexicographic Inference

The definition of lexicographic inference is based on the ordering $<^{lex}_\Delta$ induced by every belief base Δ. Here we use the notation used in (Komo and Beierle 2022) and in Definition 7.

Definition 9 ($<^{lex}_\Delta$, lexicographic inference (Lehmann 1995))**.** *The* lexicographic ordering *on vectors in $\mathbb{N}^n$ is defined by $(v_1, \ldots, v_n) <^{lex} (w_1, \ldots, w_n)$ iff there is a $k \in \{1, \ldots, n\}$ such that $v_k < w_k$ and $v_j = w_j$ for $j = k+1, \ldots, n$.*

The binary relation $\leqslant^{lex}_\Delta \subseteq \Omega \times \Omega$ on worlds induced by a belief base Δ with $EZP(\Delta) = (\Delta^0, \ldots, \Delta^k, \Delta^\infty)$ is defined by, for any $\omega, \omega' \in \Omega$,

$$\omega \leqslant^{lex}_\Delta \omega' \quad \textit{iff} \quad (|\xi^1_\Delta(\omega)|, \ldots, |\xi^k_\Delta(\omega)|, |\xi^\infty_\Delta(\omega)|) \leqslant^{lex} (|\xi^1_\Delta(\omega')|, \ldots, |\xi^k_\Delta(\omega')|, |\xi^\infty_\Delta(\omega')|).$$

Lexicographic inference $\mathrel{|\!\sim}^{lex}_\Delta$ *is induced by $<^{lex}_\Delta$: For formulas F, G, A, B, we have:*

$$F <^{lex}_\Delta G \quad \textit{iff} \quad \min(\Omega_F, <^{lex}_\Delta) <^{lex}_\Delta \min(\Omega_G, <^{lex}_\Delta)$$

$$A \mathrel{|\!\sim}^{lex}_\Delta B \quad \textit{iff} \quad AB <^{lex}_\Delta A\overline{B}$$

Note that lexicographic inference as defined by (Lehmann 1995) and presented in Definition 9 does not comply with (DI) and therefore is not an inductive inference operator if we also allow belief bases that are not strongly consistent.

Proposition 6. *Lexicographic inference violates (DI) for some weakly consistent belief bases.*

Proof. Towards a contradiction, assume that lexicographic inference satisfies (DI). Consider the belief base $\Delta = \{(b|a), (\overline{b}|a)\}$. Because of (DI), we have $a \mathrel{|\!\sim}^{lex}_\Delta b$ and $a \mathrel{|\!\sim}^{lex}_\Delta \overline{b}$. Using the definition of $\mathrel{|\!\sim}^{lex}_\Delta$, we have that $a \mathrel{|\!\sim}^{lex}_\Delta b$ implies $ab <^{lex}_\Delta a\overline{b}$, and $a \mathrel{|\!\sim}^{lex}_\Delta \overline{b}$ implies $a\overline{b} <^{lex}_\Delta ab$. This is a contradiction because $\leqslant^{lex}_\Delta$ is a total preorder. □

The following slightly adapted version of lexicographic inference does comply with (DI) and is an inductive inference operator.

Definition 10 (adapted lexicographic inference)**.** *For formulas A, B,* adapted lexicographic inference $\mathrel{|\!\sim}^{alex}_\Delta$ *is defined as*

$$A \mathrel{|\!\sim}^{alex}_\Delta B \quad \textit{iff} \quad \xi^\infty(\omega) \neq \emptyset \textit{ for all } \omega \in \Omega_A, \quad \textit{or} \quad A \mathrel{|\!\sim}^{lex}_\Delta B$$

Proposition 7. *Let Δ be a belief base and $A \in \mathcal{L}$. If $A \mathrel{|\!\not\sim}^{p}_\Delta \bot$, then $A \mathrel{|\!\sim}^{lex}_\Delta B$ iff $A \mathrel{|\!\sim}^{alex}_\Delta B$.*

Also, for any strongly consistent belief base Δ lexicographic inference and adapted lexicographic inference coincide as we then have $A \mathrel{|\!\not\sim}^{p}_\Delta \bot$ for any $A \not\equiv \bot$.

Now we are ready to connect (adapted) lexicographic inference and system W$^+$.

Lemma 3. *Let Δ be a belief base and ω, ω' be worlds. Then $\omega <^{\mathsf{w}+}_\Delta \omega'$ implies $\omega <^{lex}_\Delta \omega'$.*

Using Lemma 3, we can show that every system-W$^+$ entailment is also an entailment for adapted lexicographic inference.

Proposition 8. *Adapted lexicographic inference captures system W$^+$, i.e., for a belief base Δ and formulas A, B it holds that if $A \mathrel{|\!\sim}^{\mathsf{w}+}_\Delta B$ then $A \mathrel{|\!\sim}^{alex}_\Delta B$.*

5.3 Relation to Multipreference Closure

In (Haldimann and Beierle 2022a) it was shown that system W coincides with multipreference closure (MP-closure) (Giordano and Gliozzi 2021) for strongly consistent belief bases. Now we extend this result and show that system W^+ and MP-closure coincide for all belief bases.

Let us first recall the definition of MP-closure.

Definition 11 (exceptionality of a formula/conditional (Lehmann and Magidor 1992)). *A formula $A \in \mathcal{L}_\Sigma$ is* exceptional *for a belief base Δ if $\top \mid\!\sim^p_\Delta \neg A$. A conditional $(B|A)$ is* exceptional *for Δ if A is exceptional for Δ. The set of exceptional conditionals for Δ is denoted as $E(\Delta)$.*

Definition 12 (rank of a formula/conditional (Lehmann and Magidor 1992), order of a belief base (Giordano and Gliozzi 2021)). *Let Δ be belief base. We define a sequence of sets $C_0, C_1, \ldots$ by $C_0 = \Delta$ and $C_i = E(C_{i-1})$ for $i > 0$. The least finite l with $C_l = C_{l+1}$ is called the* order *of Δ.*

The rank *of a formula A (with respect to Δ) is the smallest i such that A is not exceptional for C_i. If A is exceptional for all C_i it has rank ∞. The rank of a conditional is the rank of its antecedence.*

Note that a belief base Δ with order l does not contain conditionals with rank l. For $l > 0$ the highest finite rank of a conditional in Δ is $l - 1$.

Definition 13 (MP-seriousness ordering $\prec^{MP}_\Delta$ (Giordano and Gliozzi 2021)). *Let Δ be a belief base with order l. For $X \subseteq \Delta$ let $(X_\infty, X_l, \ldots, X_0)_X$ be a tuple of sets such that X_i is the set of conditionals in X with rank i.*
For two tuples $(X_n, \ldots, X_1)$ and $(Y_n, \ldots, Y_1)$ we define

$$\begin{aligned} (X_1) \ll (Y_1) \quad & \text{iff} \quad X_1 \subsetneq Y_1 \\ (X_i, \ldots, X_1) \ll (Y_i, \ldots, Y_1) \quad & \text{iff} \quad X_i \subsetneq Y_i \text{ or} \\ & \quad\;\; X_i = Y_i \text{ and } (X_{i-1} \ll Y_{i-1}). \end{aligned}$$

The MP-seriousness ordering *$\prec^{MP}_\Delta$ on subsets of Δ is defined by*

$$X \prec^{MP}_\Delta Y \text{ iff } (X^\infty, X^l, \ldots, X^0)_X \ll (Y^\infty, Y^l, \ldots, Y^0)_Y.$$

MP-closure is defined in terms of MP-bases in (Giordano and Gliozzi 2021).

Definition 14 (MP-basis (Giordano and Gliozzi 2021)). *Let Δ be a belief base. Let $A \in \mathcal{L}_\Sigma$ be a formula with finite rank with respect to Δ. A set $D \subseteq \Delta$ is an* MP-basis *for A if*

- *A is consistent with $\tilde{D} = \{B \to C \mid (C|B) \in D\}$, and*
- *D is maximal with respect to the MP-seriousness ordering among the subsets of Δ with this property.*

Definition 15 (MP-closure (Giordano and Gliozzi 2021)). *Let Δ be a belief base. $A \mid\!\sim^{MP}_\Delta B$ is in the MP-closure $MP(\Delta)$ of Δ if for all MP-bases D of A it holds that $\tilde{D} \cup \{A\} \models B$.*

This definition for MP-closure resembles the definition of lexicographic inference in (Lehmann 1995); but MP-closure utilizes the MP-ordering $\prec^{MP}_\Delta$ instead of the seriousness ordering defined by Lehmann, and MP-bases only exist for formulas with finite rank.

MP-closure can be characterized utilizing certain preferential models called *MP-models*. These are defined using the following functor $\mathcal{F}_\Delta$.

Definition 16 (functor $\mathcal{F}_\Delta$ (Giordano and Gliozzi 2021)). *Let Δ be a belief base. The functor $\mathcal{F}_\Delta$ is a mapping from minimal[1] canonical ranked models of Δ to preferential models defined by*

$$\mathcal{F}_\Delta(\langle S, l, \prec\rangle) = \langle S, l, \prec_F\rangle$$

with $s \prec_F t$ iff $\xi(s) \prec^{MP}_\Delta \xi(t)$ for $s, t \in S$.

$\mathcal{F}_\Delta$ is extended to sets P of minimal canonical ranked models of Δ by $\mathcal{F}_\Delta(P) = \{\mathcal{F}_\Delta(\mathcal{M}) \mid \mathcal{M} \in P\}$.

Definition 17 (MP-model (Giordano and Gliozzi 2021)). *Let Δ be a belief base. An MP-model of Δ is any model in $\mathcal{F}_\Delta(Min_{RC}(\Delta))$.*

Proposition 9 (MP-closure representation theorem (Giordano and Gliozzi 2021)). *Let Δ be a belief base. A conditional $(B|A)$ is accepted by every MP-model of Δ iff $A \mid\!\sim^{MP}_\Delta B$.*

The MP-closure representation uses skeptical inference over all MP-models of a belief base. Giordano and Gliozzi (Giordano and Gliozzi 2021) showed that all MP-models induce the same inference relation, i.e., for any two MP-models $\mathcal{N}, \mathcal{N}'$ of a belief base Δ and any $A, B \in \mathcal{L}_\Sigma$ we have $A \mid\!\sim_{\mathcal{N}} B$ iff $A \mid\!\sim_{\mathcal{N}'} B$.

The proof that system W^+ and MP-closure coincide uses the characterization of MP-closure with MP-models.

Proposition 10. *For a belief base Δ the system-W^+ preferential model $\mathcal{M}^{\mathsf{w}}(\Delta)$ is an MP-model of Δ.*

Using this, we can show that the MP-closure of Δ coincides with the inference relation induced by $\mathcal{M}^{\mathsf{w}}(\Delta)$. This entails that the MP-closure of Δ coincides with system-W^+ inference from Δ.

Proposition 11. *For every consistent belief base Δ and formulas $A, B \in \mathcal{L}_\Sigma$ it holds that:*

- *$A \mid\!\sim^{MP}_\Delta B$ iff $A \mid\!\sim_{\mathcal{M}^{\mathsf{w}}(\Delta)} B$.*
- *$A \mid\!\sim^{MP}_\Delta B$ iff $A \mid\!\sim^{\mathsf{w}+}_\Delta B$.*

6 System W^+ Satisfies Syntax Splitting

Initially defined for belief sets (Parikh 1999), the notion of syntax splitting was transferred to many other belief representation frameworks, including belief bases. Kern-Isberner, Beierle, and Brewka formulated postulates for inductive inference operators, (Rel) and (Ind), that guide the behaviour of the inference for belief bases with syntax splitting (Kern-Isberner, Beierle, and Brewka 2020). The postulate (SynSplit) is the combination of (Rel) and (Ind). In this section, we will first adapt these syntax splitting postulates. Then we will formulate properties of SPO-based inductive inference operators that imply the satisfaction of the syntax splitting postulates.

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[1]This means minimality w.r.t. $<_{FIMS}$, an SPO on ranked models. The definition of $<_{FIMS}$ can be found in (Giordano and Gliozzi 2021); here it is only important that $(S, l, \prec) <_{FIMS} (S', l', \prec')$ implies that $S = S'$ and $l = l'$.

Definition 18 (syntax splitting for belief bases (adapted from (Kern-Isberner, Beierle, and Brewka 2020)))**.** *Let Δ be a belief base over a signature Σ. A partitioning $\{\Sigma_1, \dots, \Sigma_n\}$ of Σ is a* syntax splitting *for Δ if there is a partitioning $\{\Delta_1, \dots, \Delta_n\}$ of Δ such that $\Delta_i \subseteq (\mathcal{L}|\mathcal{L})_{\Sigma_i}$ for every $i = 1, \dots, n$.*

In this paper, we focus on syntax splittings $\{\Sigma_1, \Sigma_2\}$ of Δ with size two. A syntax splitting $\{\Sigma_1, \Sigma_2\}$ with the corresponding partition $\{\Delta_1, \Delta_2\}$ of Δ is denoted as $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$. Results for belief bases with syntax splittings in more than two parts can be obtained by iteratively applying the postulates and constructions presented here.

While the syntax splitting postulates in (Kern-Isberner, Beierle, and Brewka 2020) take only strongly consistent belief bases into account, we now extend these postulates to also cover weakly consistent belief bases.

Postulate (Rel$^+$), (Ind$^+$), (SynSplit$^+$). *An inductive inference operator $C : \Delta \mapsto \mathrel{|\!\sim}_\Delta$ satisfies*

(Rel$^+$) *if for any weakly consistent $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_{\Sigma_i}$ for $i = 1, 2$ we have that*

$$A \mathrel{|\!\sim}_\Delta B \quad \textit{iff} \quad A \mathrel{|\!\sim}_{\Delta_i} B. \tag{3}$$

(Ind$^+$) *if for any weakly consistent $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_{\Sigma_i}$, $D \in \mathcal{L}_{\Sigma_j}$ for $i, j \in \{1, 2\}, i \neq j$ such that $D \not\mathrel{|\!\sim}^p_\Delta \bot$, we have*

$$A \mathrel{|\!\sim}_\Delta B \quad \textit{iff} \quad AD \mathrel{|\!\sim}_\Delta B. \tag{4}$$

(SynSplit$^+$) *if it satisfies both (Rel$^+$) and (Ind$^+$).*

Inductive inference operators can not only be defined by directly giving the mapping from belief bases to inference relations. Another way is to define inductive inference operators using SPOs as an intermediate step: we define a mapping from belief bases to SPOs and then obtain the inference relation from the SPO by $A \mathrel{|\!\sim}_\prec B$ iff $AB \prec A\overline{B}$ with

$$A \prec B \quad \text{iff} \quad \text{for every } \omega' \in \Omega_B \text{ there is an } \omega \in \Omega_A \text{ such that } \omega \prec \omega'.$$

This way we can define an *SPO-based inductive inference operator* to be a mapping $C^{spo} : \Delta \mapsto \prec_\Delta$ that maps a belief base to a SPO $\prec_\Delta$ such that $\prec_\Delta \models \Delta$ and $\prec_\emptyset = \emptyset$.

Based on (Rel^{spo}) and (Ind^{spo}) (Haldimann and Beierle 2022b) we introduce the properties **(Rel^{spo+})** and **(Ind^{spo+})** that ensure compliance with (Rel$^+$) and (Ind$^+$) also for belief bases that force some worlds to be infeasible.

Postulate (Rel^{spo+}), (Ind^{spo+}). *An SPO-based inductive inference operator $C^{spo} : \Delta \mapsto \prec_\Delta$ satisfies*

(Rel^{spo+}) *if for any weakly consistent $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B, F \in \mathcal{L}_{\Sigma_i}$ for $i \in \{1, 2\}$, such that $A \not\mathrel{|\!\sim}^p_\Delta \bot$ and $B \not\mathrel{|\!\sim}^p_\Delta \bot$ it holds that*

$$F \textit{ is feasible in } \prec_\Delta \quad \textit{iff} \quad F \textit{ is feasible in } \prec_{\Delta_i} \quad \textit{and}$$
$$A \prec_\Delta B \quad \textit{iff} \quad A \prec_{\Delta_i} B.$$

(Ind^{spo+}) *if for any weakly consistent $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B, F \in \mathcal{L}_{\Sigma_i}$, $D \in \mathcal{L}_{\Sigma_j}$ for $i, j \in \{1, 2\}$, $i \neq j$, such that $A \not\mathrel{|\!\sim}^p_\Delta \bot$, $B \not\mathrel{|\!\sim}^p_\Delta \bot$, and $D \not\mathrel{|\!\sim}^p_\Delta \bot$, it holds that*

$$F \textit{ is feasible in } \prec_\Delta \quad \textit{iff} \quad FD \textit{ is feasible in } \prec_\Delta \quad \textit{and}$$
$$A \prec_\Delta B \quad \textit{iff} \quad AD \prec_\Delta BD.$$

Proposition 12. *Let C^{spo} be an SPO-based inductive inference operator. If C^{spo} satisfies (Rel^{spo+}) then it satisfies (Rel$^+$). If C^{spo} satisfies (Ind^{spo+}) then it satisfies (Ind$^+$).*

To show that system W$^+$ satisfies (Rel^{spo+}) and (Ind^{spo+}), we first have to show how a syntax splitting on a belief base carries over to the tolerance partition. The main result is that the tolerance partition of a belief base $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$ is the element-wise conjunction of the tolerance partitions of Δ_1 and Δ_2, i.e., $\Delta_i^j = \Delta^j \cap \Delta_i$. Based on that, we can show properties of the preferred structure on worlds induced by a belief base with syntax splitting: For a weakly consistent $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$ and feasible worlds $\omega, \omega', \omega_a, \omega_b$ we have that:

- If $\omega <^{\mathsf{w}+}_\Delta \omega'$, then $\omega <^{\mathsf{w}+}_{\Delta_1} \omega'$ or $\omega <^{\mathsf{w}+}_{\Delta_2} \omega'$.
- If $\omega <^{\mathsf{w}+}_{\Delta_1} \omega'$ and $\omega_{|\Sigma_2} = \omega'_{|\Sigma_2}$, then $\omega <^{\mathsf{w}+}_\Delta \omega'$.
- If $\omega^a{}_{|\Sigma_1} = \omega^b{}_{|\Sigma_1}$, then it is $\omega^a <^{\mathsf{w}+}_{\Delta_1} \omega'$ iff $\omega^b <^{\mathsf{w}+}_{\Delta_1} \omega'$.

Using these results, we can show that system W$^+$ fulfils both (Rel^{spo+}) and (Ind^{spo+}).

Proposition 13. *System W$^+$ fulfils (Rel^{spo+}).*

Proposition 14. *System W$^+$ fulfils (Ind^{spo+}).*

Combining Propositions 13 and 14 yields that system W$^+$ fulfils (SynSplit$^+$).

Proposition 15. *System W$^+$ satisfies (SynSplit$^+$).*

7 Conclusions and Future Work

In this paper, we extended the definition of system W to also cover belief bases that are weakly consistent. After defining the system W$^+$, we re-established the relations of system W$^+$ to system Z/rational closure, lexicographic inference, and MP-closure. To do this, we had to slightly adapt the definition of lexicographic inference for it to be an inductive inference operator. Furthermore, we generalized the postulates (Rel), (Ind), and (SynSplit) to also be applicable to weakly consistent belief bases and showed that system W$^+$ satisfies syntax splitting.

Future work includes showing that system W$^+$ also satisfies the more general conditional syntax splitting postulates introduced in (Heyninck et al. 2023). Additionally, we will extend c-representations (Kern-Isberner 2001; 2001) and c-inference (Beierle, Eichhorn, and Kern-Isberner 2016) to weakly consistent belief bases and investigate the relation between system W$^+$ and the thus extended c-inference.

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