Normal Forms of Conditional Belief Bases Respecting Inductive Inference

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Abstract

Normal forms of syntactic entities play an important role in many different areas in computer science. In this paper, we address the question of how to obtain normal forms and minimal normal forms of conditional belief bases in order to, e.g., ease reasoning with them or to simplify their comparison. We introduce notions of equivalence of belief bases taking nonmonotonic inductive inference operators into account. Furthermore, we also consider renamings of belief bases induced by renamings of the underlying signatures. We show how renamings constitute another dimension of normal forms. Based on these different dimensions, we introduce and illustrate various useful normal forms and show their properties, advantages, and interrelationships.

1 Introduction

Conditional belief bases consisting of conditionals of the form *"If A then usually B"* are commonly used to represent and reason with beliefs. Various semantics have been proposed for conditionals, e.g., (Benferhat, Dubois, and Prade 1999; Spohn 2012; Kern-Isberner 2001; Beierle and Kern-Isberner 2012). Generally, the inference properties of the semantics have been in the focus of the research e.g. (Adams 1965; Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992), less attention has been paid do the study of normal form for conditional belief bases, e.g. (Beierle and Kutsch 2019a; Beierle 2019; Beierle and Haldimann 2020). In this paper, we investigate normal forms of belief bases in particular from the viewpoint obtained by respecting inference methods satisfying corresponding properties. We introduce notions of equivalence of belief bases taking inductive inference operators into account, leading to various normal forms and to unique minimal normal forms. Orthogonal to this dimension, we employ signature renamings and show how they can be combined systematically with other normal forms. We investigate the properties of the introduced normal forms and their interrelationships and present observations from our empirical evaluation of normal forms that support our formal investigations.

2 Background: Conditional Logic

Let $\mathcal{L}(\Sigma)$, or just $\mathcal{L}$, be the propositional language over a finite signature Σ. We call a signature Σ with a linear ordering $\lessdot$ an *ordered signature* and denote it by $(\Sigma, \lessdot)$. For $A, B \in \mathcal{L}$, we write AB for $A \wedge B$ and $\overline{A}$ for $\neg A$. We identify the set of all complete conjunctions over Σ with the set Ω of possible worlds over $\mathcal{L}$. For $\omega \in \Omega$ and $A \in \mathcal{L}$, $\omega \models A$ means that A holds in ω. Two formulas A, B are *equivalent*, denoted as $A \equiv B$, if $\Omega_A = \Omega_B$, with $\Omega_A = \{\omega \mid \omega \models A\}$.

We define the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. The intuition of a conditional $(B|A)$ is that if A holds then usually B holds, too. As semantics for conditionals, we use functions $\kappa : \Omega \to \mathbb{N}$ such that $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$, called *ordinal conditional functions (OCF)*, introduced (in a more general form) by Spohn. They express degrees of plausibility where a lower degree denotes "less surprising". Each κ uniquely extends to a function $\kappa : \mathcal{L} \to \mathbb{N} \cup \{\infty\}$ with $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$ where $\min \emptyset = \infty$. An OCF κ *accepts* a conditional $(B|A)$, written $\kappa \models (B|A)$, if $\kappa(AB) < \kappa(A\overline{B})$. A conditional $(B|A)$ is trivial if it is *self-fulfilling* ($A \models B$) or *contradictory* ($A \models \overline{B}$). We say that $(B|A)$ and $(B'|A')$ are *conditionally equivalent*, denoted by $(B|A) \equiv_{ce} (B'|A')$, if $A \equiv A'$ and $AB \equiv A'B'$. A finite set $\mathcal{R} \subseteq (\mathcal{L}|\mathcal{L})$ is a *belief base*. An OCF κ accepts $\mathcal{R}$ if κ accepts all conditionals in $\mathcal{R}$, and $\mathcal{R}$ is *consistent* if an OCF accepting $\mathcal{R}$ exists.

For orderings like $\leqslant$ or $\preceq$ the strict variants are denoted by $<$ or $\prec$, respectively, i.e., $a < b$ iff $a \leqslant b$ and $b \nleqslant a$.

3 Inductive Inference Operators

The notion of *inductive inference operator* formalizes how an inference relation $\mid\!\sim \; \subset \mathcal{L} \times \mathcal{L}$ is obtained by inductive completion of a given belief base.

Definition 1 (inductive inference operator (Kern-Isberner, Beierle, and Brewka 2020))**.** *An* inductive inference operator *is a mapping* $C\colon \Delta \mapsto \;\mid\!\sim_\Delta$ *that maps a belief base to an inference relation such that direct inference (DI) and trivial vacuity (TV) are fulfilled:*

(DI) *if* $(B|A) \in \Delta$ *then* $A \mid\!\sim_\Delta B$

(TV) *if* $\Delta = \emptyset$ *and* $A \mid\!\sim_\Delta B$ *then* $A \models B$

If no confusion arises, we will often simply use $\mid\!\sim$ to denote the inductive inference operator mapping Δ to $\mid\!\sim_\Delta$. Examples of inductive inference operators are:

p-entailment $\mid\!\sim^p$ (Goldszmidt and Pearl 1996) considers all ranking models and coincides with system P-inference

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(Lehmann and Magidor 1992; Dubois and Prade 1994).

system Z $\mid\!\sim^{z}$ (Goldszmidt and Pearl 1996) uses the inclusion maximal tolerance partition of Δ and it coincides with rational closure (Lehmann and Magidor 1992).

c-inference $\mid\!\sim^{c}$ (Beierle et al. 2018; 2021) considers all c-representations (Kern-Isberner 2004).

system W $\mid\!\sim^{\mathsf{w}}$ (Komo and Beierle 2022) captures both c-inference and system Z and thus rational closure.

In the following, we formalize some properties an inductive inference operator can have: (AND) and right weakening (RW) from system P, self-fulfilling (SF), semi-monotony (SM) (Reiter 1980; Goldszmidt and Pearl 1996), syntax independence (SI), and conditional equivalence (CE).

(AND) $A \mid\!\sim B$ and $A \mid\!\sim C$ imply $A \mid\!\sim B \wedge C$

(RW) $B \models C$ and $A \mid\!\sim B$ imply $A \mid\!\sim C$

(SF) $A \models B$ implies $\mid\!\sim_{\Delta \cup \{(B|A)\}} = \mid\!\sim_{\Delta}$

(SM) $\Delta \subseteq \Delta'$ and $A \mid\!\sim_{\Delta} B$ imply $A \mid\!\sim_{\Delta'} B$

(SI) $A \equiv A'$ and $B \equiv B'$ imply $\mid\!\sim_{\Delta \cup \{(B|A)\}} = \mid\!\sim_{\Delta \cup \{(B'|A')\}}$

(CE) $(B|A) \equiv_{ce} (B'|A')$ imp. $\mid\!\sim_{\Delta \cup \{(B|A)\}} = \mid\!\sim_{\Delta \cup \{(B'|A')\}}$

4 CDNF and Normal Form Conditionals

We can abstract from the syntactic variants of the underlying propositional language $\mathcal{L}$ and represent each formula $A \in \mathcal{L}$ uniquely by its set Ω_A of satisfying worlds, called *canonical disjunctive normal form (CDNF)* of A.

Definition 2 (CDNF, CDNF($\mathcal{R}$)). *A belief base $\mathcal{R}$ over Σ using the set-oriented representation of CDNF for all antecedents and consequents is in CDNF. The CDNF of a belief base $\mathcal{R}$ is CDNF($\mathcal{R}$) $= \{(\Omega_B|\Omega_A) \mid (B|A) \in \mathcal{R}\}$.*

For comparing belief bases, an important criterion is whether they induce the same entailments.

Definition 3 ($\equiv_{\mid\sim}$, inferentially equivalent with respect to $\mid\!\sim$). *Two belief bases $\mathcal{R}, \mathcal{R}'$ are* inferentially equivalent with respect to $\mid\!\sim$*, denoted by $\mathcal{R} \equiv_{\mid\sim} \mathcal{R}'$, if, for all formulas A, B, $A \mid\!\sim_{\mathcal{R}} B$ holds if and only if $A \mid\!\sim_{\mathcal{R}'} B$.*

A desirable property for a normal form *<NF>* is that it also covers all sets of entailments that can be obtained from a belief base not in *<NF>*. In the following, we will use $\mathbf{\Delta}$(*<NF>*) to denote the set of all belief bases in *<NF>*.

Definition 4 ($\mid\!\sim$-complete). *$IR \subseteq \mathcal{L} \times \mathcal{L}$ is a* $\mid\!\sim$-relation *if there is a consistent belief base $\mathcal{R}$ with $\mid\!\sim_{\mathcal{R}} = IR$. A set $\mathcal{S}$ of belief bases is* $\mid\!\sim$-complete *if for every $\mid\!\sim$-relation IR there is $\mathcal{R} \in \mathcal{S}$ with $\mid\!\sim_{\mathcal{R}} = IR$. A normal form <NF> is* $\mid\!\sim$-complete *if $\mathbf{\Delta}$(<NF>) is $\mid\!\sim$-complete.*

Proposition 5. *CDNF is $\mid\!\sim$-complete if $\mid\!\sim$ satisfies (SI).*

Observe that $\mid\!\sim^{p}$, $\mid\!\sim^{z}$, $\mid\!\sim^{c}$, and $\mid\!\sim^{\mathsf{w}}$are ignorant with respect to self-fulfilling conditionals; furthermore, they treat two conditionals having the same verification and the same falsification behaviour identically. In the following proposition, the two conditions $B \subsetneqq A$ and $B \neq \emptyset$ ensure the falsifiability and the verifiability of a conditional $(B|A)$, thereby excluding any trivial conditional.

Proposition 6 ($NFC(\Sigma)$ (Beierle and Kutsch 2019b)). *For $NFC(\Sigma) = \{(B|A) \mid A \subseteq \Omega_{\Sigma},\ B \subsetneqq A,\ B \neq \emptyset\}$, the set of* normal form conditionals *over Σ, the following holds: (i) $NFC(\Sigma)$ does not contain any trivial conditional. (ii) For every nontrivial conditional over Σ there is a conditionally equivalent conditional in $NFC(\Sigma)$. (iii) All conditionals in $NFC(\Sigma)$ are pairwise not conditionally equivalent.*

Example 7. *Using first the CDNF for $\{(\overline{a}|b), (b|a \vee b), (\overline{a} \vee b|a \vee \overline{b})\}$ and then replacing every conditional by its equivalent normal form conditional yields $\{(\{\overline{a}b\}|\{ab, \overline{a}b\}), (\{ab, \overline{a}\overline{b}\}|\{ab, a\overline{b}, \overline{a}\overline{b}\}), (\{ab, \overline{a}\overline{b}\}|\{ab, a\overline{b}, \overline{a}\overline{b}\})\}$.*

Using only $NFC(\Sigma)$-conditionals yields the CNF normal form, and for each $\mathcal{R}$, there is a uniquely determined CNF.

Definition 8 (CNF, CNF($\mathcal{R}$)). *A belief base $\mathcal{R}$ over Σ is in* conditional normal form *(CNF) if $\mathcal{R} \subseteq NFC(\Sigma)$. For each consistent belief base $\mathcal{R}$ over Σ, its CNF representation is CNF($\mathcal{R}$) $= \{(\Omega_{AB}|\Omega_A) \mid (B|A) \in \mathcal{R}\} \cap NFC(\Sigma)$.*

Proposition 9. *CNF is $\mid\!\sim$-complete if $\mid\!\sim$ satisfies (SF) and (CE).*

Note that Prop. 9 covers all inductive inference operators discussed above, in particular, $\mid\!\sim^{p}$, $\mid\!\sim^{z}$, $\mid\!\sim^{c}$, and $\mid\!\sim^{\mathsf{w}}$.

5 Antecedent Normal Form

The basic idea of antecedentwise equivalence of two belief bases $\mathcal{R}, \mathcal{R}'$ is to require that the sets of conditionals having equivalent antecedents correspond to each other in $\mathcal{R}$ and $\mathcal{R}'$ (Beierle and Kutsch 2019a).

Definition 10 (ANF (Beierle and Kutsch 2019a)). *Let $\mathcal{R}$ be a consistent belief base. $Ant(\mathcal{R}) = \{A \mid (B|A) \in \mathcal{R}\}$ are the* antecedents of $\mathcal{R}$*, and for $A \in Ant(\mathcal{R})$, the set $\mathcal{R}_{|A} = \{(B'|A') \mid (B'|A') \in \mathcal{R}$ and $A \equiv A'\}$ is the set of A-*conditionals in $\mathcal{R}$. *$\mathcal{R}$ is in* antecedent normal form (ANF) *if it is in CNF and $|\mathcal{R}_{|A}| = 1$ for all $A \in Ant(\mathcal{R})$.*

For each belief base there is a uniquely determined ANF.

Proposition 11 (ANF($\mathcal{R}$)). *If $\mathcal{R}$ is a consistent belief base, then ANF($\mathcal{R}$) $= \{(\Omega_{AB_1 \ldots B_n}|\Omega_A) \mid A \in Ant(\mathcal{R}), \mathcal{R}_{|A} = \{(B_1|A_1), \ldots, (B_n|A_n)\}, A \not\models B_1 \ldots B_n\}$ is in ANF.*

If $\mid\!\sim$ satisfies (AND) and (RW), $(B|A)$ and $(B'|A)$ on the one hand and $(BB'|A)$ on the other hand can be derived from each other. If $\mid\!\sim$ also satisfies (SM) then it does not matter whether a belief base contains $(B|A)$ and $(B'|A)$, or $(BB'|A)$, yielding the following proposition.

Proposition 12 (ANF($\mathcal{R}$)). *Let $\mathcal{R}$ be a consistent belief base. Then $\mathcal{R} \equiv_{\mid\sim}$ ANF($\mathcal{R}$) if $\mid\!\sim$ satisfies (SF), (CE), (AND), (RW), and (SM).*

A consequence of Proposition 12 we get:

Proposition 13. *ANF is $\mid\!\sim$-complete if $\mid\!\sim$ satisfies (SF), (CE), (AND), (RW), and (SM).*

Thus, because $\mid\!\sim^{p}$ satisfies (SF), (CE), (AND), (RW), and (SM), ANF is $\mid\!\sim^{p}$-complete and $\mathcal{R} \equiv_{\mid\sim^{p}}$ ANF($\mathcal{R}$).

Observation 1. *While all $\mid\!\sim \in \{\mid\!\sim^{z}, \mid\!\sim^{c}, \mid\!\sim^{\mathsf{w}}\}$ satisfy (SF), (CE), (AND), and (RW), they fail to satisfy (SM). However, empirical evidence obtained from using InfOCF*

(Kutsch and Beierle 2021) supports the conjecture that ANF is also $\mid\!\sim$-complete for system Z, c-inference, and system W. A systematic generation of belief bases over $\Sigma_{ab} = \{a, b\}$ using the approach given in (Beierle and Haldimann 2020) and a comparison with respect to $\equiv_{\mid\sim}$ suggets that for $\mid\!\sim\ \in \{\mid\!\sim^z, \mid\!\sim^c, \mid\!\sim^{\mathrm{w}}\}$ and all $\mathcal{R}$ over Σ_{ab} the inference relation $\mid\!\sim_{\mathcal{R}}$ can already be obtained from a belief base in ANF.

6 Reduced Antecedent Normal Form

A belief base in ANF may still contain redundancies in form of conditionals that can be inferred form the other conditionals in $\mathcal{R}$. For instance, in $\mathcal{R} = \{(ab|a), (ab|b), (ab|a \vee b)\}$, the third conditional can be derived from the first two conditionals with system P axiom (OR); omitting it does not change the induced inference relation of $\mathcal{R}$ with respect to system P inference. The reduced ANF (Beierle and Haldimann 2020) avoids such redundancies with respect to system P inference. Here, we generalize this concept by taking any inductive inference operator into account.

Definition 14 ($\mid\!\sim$-reduced, RANF$_{\mid\sim}$)**.** *A belief base $\mathcal{R}$ is* $\mid\!\sim$-reduced *if there is no conditional $(B|A) \in \mathcal{R}$ such that $A \mid\!\sim_{\mathcal{R}\setminus(B|A)} B$. $\mathcal{R}$ is in* $\mid\!\sim$-reduced *antecedent normal form (in RANF$_{\mid\sim}$) if $\mathcal{R}$ is $\mid\!\sim$-reduced and in ANF.*

In general, for an inductive inference operator $\mid\!\sim$ and a belief base $\mathcal{R}$ there may be several $\mathcal{R}', \mathcal{R}''$ in RANF$_{\mid\sim}$ with $\mathcal{R} \equiv_{\mid\sim} \mathcal{R}'$ and $\mathcal{R} \equiv_{\mid\sim} \mathcal{R}''$, but $\mathcal{R}' \neq \mathcal{R}''$. Thus, in contrast to the CDNF, CNF, and ANF normal forms, there is not a unique RANF$_{\mid\sim}$ for every a belief base.

Definition 15 ($\mathcal{RANF}_{\mid\sim}(\mathcal{R})$)**.** *The set of RANF$_{\mid\sim}$ representations of $\mathcal{R}$, denoted by $\mathcal{RANF}_{\mid\sim}(\mathcal{R})$, is given by $\mathcal{RANF}_{\mid\sim}(\mathcal{R}) = \{\mathcal{R}' \mid \mathcal{R}' \equiv_{\mid\sim} \mathcal{R}, \mathcal{R}'$ is in RANF$_{\mid\sim}\}$.*

For instance, the non-deterministic transformation system Θ^{ra} provided in (Beierle and Haldimann 2020) takes system P inference into account and ensures that every $\mathcal{R}' \in \Theta^{ra}(\mathcal{R})$ is in RANF$_{\mid\sim^p}$ and $\mathcal{R} \equiv_{\mid\sim^p} \mathcal{R}'$. But not every belief base in $\mathcal{RANF}_{\mid\sim^p}(\mathcal{R})$ is in $\Theta^{ra}(\mathcal{R})$.

Example 16. *For a shorter and more concise notation of formulas in CDNF we use $\nu(F)$ to denote the CDNF of a formula F in this example; e.g., for $\Sigma = \{a, b, c, d\}$, we have $\nu(abc) = CDNF(abc) = \{abcd, abc\overline{d}\}$. Consider the belief bases $\mathcal{R} = \{(\nu(ab)|\nu(a)), (\nu(ab)|\nu(b)), (\nu((a \vee c)d)|\nu(a \vee c))\}$ and $\mathcal{R}' = \{(\nu(ab)|\nu(a)), (\nu(ab)|\nu(b)), (\nu((b \vee c)d)|\nu(b \vee c))\}$. We have $\mathcal{R} \equiv_{\mid\sim^p} \mathcal{R}'$ and $\mathcal{R}'$ is in RANF, and thus $\mathcal{R}' \in \mathcal{RANF}_{\mid\sim}(\mathcal{R})$.*

The completenes property about ANF in Proposition 13 can be generalized to RANF$_{\mid\sim}$.

Proposition 17. *RANF$_{\mid\sim}$ is $\mid\!\sim$-complete if $\mid\!\sim$ satisfies (SF), (CE), (AND), (RW), and (SM).*

Thus, RANF$_{\mid\sim^p}$ is $\mid\!\sim^p$-complete.

Observation 2. *An extension of the empirical evaluation discussed in Observation 1 showed that for $\mid\!\sim\ \in \{\mid\!\sim^z, \mid\!\sim^c, \mid\!\sim^{\mathrm{w}}\}$ and all $\mathcal{R}$ in ANF over Σ_{ab}, the inference relation $\mid\!\sim_{\mathcal{R}}$ can be obtained from a belief base in RANF$_{\mid\sim}$, suggesting that RANF$_{\mid\sim}$ is $\mid\!\sim$-complete for system Z, for c-inference, and for system W.*

7 Minimal Normal Form

Here, we employ a linear ordering on the set of belief bases over $NFC(\Sigma)$ as it is developed in (Beierle and Haldimann 2020). This ordering uses signature renamings, where a function $\rho : \Sigma \to \Sigma$ is a *renaming* if ρ is a bijection. E.g., the function ρ_{ab} with $\rho_{ab}(a) = b$ and $\rho_{ab}(b) = a$ is a renaming for Σ_{ab}. As usual, ρ is extended canonically to worlds, formulas, conditionals, belief bases, and to sets thereof.

Definition 18 ($\simeq$)**.** *Let X, X' be two signatures, worlds, formulas, belief bases, sets, or relations over one of these items. We say that X and X' are* isomorphic with respect to signature renamings*, denoted by $X \simeq X'$, if there exists a renaming ρ such that $\rho(X) = X'$.*

For a set M, $m \in M$, and an equivalence relation $\equiv$ on M, the set of equivalence classes induced by $\equiv$ is denoted by $[M]_{/\equiv}$, and the unique equivalence class containing m is denoted by $[m]_{\equiv}$. E.g., $[\Omega_{\Sigma_{ab}}]_{/\simeq} = \{[ab]_{\simeq}, [a\overline{b}, \overline{a}b]_{\simeq}, [\overline{a}\overline{b}]_{\simeq}\}$ are the three equivalence classes of worlds over $\Sigma_{ab} = \{a, b\}$, and we have $[(ab|ab \vee a\overline{b})]_{\simeq} = [(ab|ab \vee \overline{a}b)]_{\simeq}$.

Based on the equivalence classes with respect to $\simeq$, the linear ordering $\prec$ on $NFC(\Sigma)$ is defined in (Beierle and Haldimann 2020) for each ordered signature Σ. We will omit the formal definition $\prec$ in this paper as it is not of importance here. The $\prec$-minimal conditional in each equivalence class in $[NFC(\Sigma_{ab})]_{/\simeq}$ is the canonical representative of that class, called *canonical normal form conditional*. We can extend $\prec$ to an ordering on belief bases.

Definition 19 ($\mathcal{R} \preccurlyeq \mathcal{R}'$ (Beierle and Haldimann 2020))**.** *The lexicographic extension of the ordering $\preccurlyeq$ on $NFC(\Sigma)$ to strings over $NFC(\Sigma)$ is denoted by $\preccurlyeq_{lex}$. For belief bases $\mathcal{R} = \{r_1, \ldots, r_n\}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_{n'}\}$ over $NFC(\Sigma)$ with $r_i \prec r_{i+1}$ and $r'_j \prec r'_{j+1}$ the ordering $\preccurlyeq_{set}$ is given by: $\mathcal{R} \preccurlyeq_{set} \mathcal{R}'$ iff $n < n'$, or $n = n'$ and $r_1 \ldots r_n \preccurlyeq_{lex} r'_1 \ldots r'_{n'}$. Furthermore, $\mathcal{R} \preccurlyeq \mathcal{R}'$ stands for $\mathcal{R} \preccurlyeq_{set} \mathcal{R}'$.*

Note that $\preccurlyeq$ is a linear ordering on belief bases.

Definition 20 (MNF$_{\mid\sim}$)**.** *A belief base $\mathcal{R}$ is in* minimal normal form with respect to $\mid\!\sim$ (in MNF$_{\mid\sim}$)*, if $\mathcal{R}$ is in CNF and for every $\mathcal{R}'$ in CNF with $\mathcal{R} \equiv_{\mid\sim} \mathcal{R}'$ it holds that $\mathcal{R} \preccurlyeq \mathcal{R}'$.*

As immediate consequence, we get the following:

Proposition 21 (MNF$_{\mid\sim}(\mathcal{R})$)**.** *For every inductive inference operator $\mid\!\sim$ and every consistent belief base $\mathcal{R}$ in CNF there is a uniquely determined belief base in MNF$_{\mid\sim}$, denoted by MNF$_{\mid\sim}(\mathcal{R})$, with $\mathcal{R} \equiv_{\mid\sim}$ MNF$_{\mid\sim}(\mathcal{R})$.*

Completeness for CNF (Prop. 9) also holds for MNF$_{\mid\sim}$.

Proposition 22. *MNF$_{\mid\sim}$ is $\mid\!\sim$-complete if $\mid\!\sim$ satisfies (SF) and (CE).*

If $\mid\!\sim$ also satisfies (AND), (RW), and (SM) then MNF$_{\mid\sim}(R)$ is among the RANF$_{\mid\sim}$ representations of $\mathcal{R}$.

Proposition 23. *If $\mathcal{R}$ is in MNF$_{\mid\sim}$ and $\mid\!\sim$ satisfies (SF), (CE), (AND), (RW), and (SM) then $\mathcal{R} \in \mathcal{RANF}_{\mid\sim}(\mathcal{R})$.*

Thus, for $\mid\!\sim$ satisfying (SF), (CE), (AND), (RW), and (SM), MNF$_{\mid\sim}$ is a refinement of RANF$_{\mid\sim}$ in the sense that $\mathbf{\Delta}$(MNF$_{\mid\sim}$) $\subseteq$ $\mathbf{\Delta}$(RANF$_{\mid\sim}$); for instance, $\mathbf{\Delta}$(MNF$_{\mid\sim^p}$) $\subseteq$ $\mathbf{\Delta}$(RANF$_{\mid\sim^p}$) holds. Furthermore, according to the study of

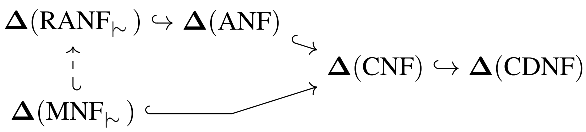

Figure 1: Overview of normal forms for conditional belief bases. Arrows indicate subset relationships. The dashed arrow holds if $\mid\!\sim$ satisfies (SF), (CE), (AND), (RW), and (SM), cf. Proposition 23.

$\mid\!\sim^p$-relations in (Beierle, Haldimann, and Kutsch 2021), we have $|\mathbf{\Delta}(\mathrm{MNF}_{\mid\sim^p})| = 485$ and $|\mathbf{\Delta}(\mathrm{RANF}_{\mid\sim^p})| = 4.168$.

An overview over the relations between the sets of all belief bases in a certain normal form is given in Figure 1.

Observation 3. *Our empirical evaluations suggest that* $MNF_{\mid\sim}$ *is* $\mid\!\sim$*-complete for system Z, for c-inference, and for system W although (SM) does not hold in these cases. Furthermore, they revealed that for* $\mid\!\sim \; \in \{\mid\!\sim^z, \mid\!\sim^c, \mid\!\sim^{\mathsf{w}}\}$*, the* $\mid\!\sim$*-relations of* Σ_{ab} *can be obtained from* $\mathbf{\Delta}(MNF_{\mid\sim^p})$ *and that* $\mathbf{\Delta}(MNF_{\mid\sim}) \subseteq \mathbf{\Delta}(MNF_{\mid\sim^p})$.

8 Normal Forms Respecting Renamings

The linear ordering $\preccurlyeq$ ensures that there is a unique *renamnig normal form* (Beierle and Haldimann 2020).

Definition 24 (ρNF, ρNF($\mathcal{R}$))**.** *A belief base* $\mathcal{R}$ *in CNF is in* renaming normal form *(ρNF) if for every* $\mathcal{R}'$ *with* $\mathcal{R} \simeq \mathcal{R}'$ *it holds that* $\mathcal{R} \preccurlyeq \mathcal{R}'$*. For every consistent* $\mathcal{R}$ *in CNF, the* renaming normal form ρNF($\mathcal{R}$) of $\mathcal{R}$ *is the uniquely determined belief base in ρNF such that* $\mathcal{R} \simeq \rho NF(\mathcal{R})$.

If *<NF>* is one of the other normal forms, we say that a belief base $\mathcal{R}$ is in *renaming <NF>*, abbreviated by ρ*<NF>*, if $\mathcal{R}$ is in ρNF and also in *<NF>*.

Proposition 25 (ρ*<NF>*($\mathcal{R}$))**.** *Let* $\mid\!\sim$ *be an inductive inference operator, and* $\mathcal{R}$ *be in CNF. For <NF>* $\in \{CNF, ANF, MNF_{\mid\sim}\}$*, the* ρ*<NF> of* $\mathcal{R}$*, denoted by* ρ*<NF>*($\mathcal{R}$)*, is uniquely determined by* ρ*<NF>*($\mathcal{R}$) $=$ $\rho NF($*<NF>*$(\mathcal{R}))$*. The set of* $\rho RANF_{\mid\sim}$ *representations of* $\mathcal{R}$*, denoted by* $\rho\mathcal{RANF}_{\mid\sim}(\mathcal{R})$*, is given by* $\rho\mathcal{RANF}_{\mid\sim}(\mathcal{R}) = \{\rho NF(\mathcal{R}') \mid \mathcal{R}' \in \mathcal{RANF}_{\mid\sim}(\mathcal{R})\}$.

When generalizing the notions of $\equiv_{\mid\sim}$ and of $\mid\!\sim$-complete (Definitions 3 and 4) by taking renamings into account, the results of Propositions 9, 13, 17, and 22 carry over to the corresponding renaming normal forms.

Observation 4. *Over the signature* Σ_{ab}*, there are 4.168 belief bases in* $RANF_{\mid\sim^p}$*. For p-entailment, we have* $|\mathbf{\Delta}(MNF_{\mid\sim^p})| = 484$ *and* $|\mathbf{\Delta}(\rho MNF_{\mid\sim^p})| = 262$ *using renamings. For system Z, we have* $|\mathbf{\Delta}(MNF_{\mid\sim^z})| = 75$ *and* $|\mathbf{\Delta}(\rho MNF_{\mid\sim^z})| = 44$.

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