A Complete Map of Conditional Knowledge Bases in Different Normal Forms and Their Induced System P Inference Relations Over Small Signatures

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Abstract

Conditional knowledge bases consisting of qualitative conditionals play a predominant role in knowledge representation and reasoning. In this paper, we develop a full map of all consistent conditional knowledge bases over a small signature in different normal forms. We introduce two new normal forms that take the induced system P inference relation into account, the *system P normal form* (SPNF) and the *renaming SPNF* (ρSPNF) considering additionally renamings of the underlying signature. For a two-element signature, we systematically generate and compare all consistent knowledge bases in ANF, RANF, SPNF, and their renaming counterparts, as well as all complete system P inference relations induced by conditional knowledge bases.

1 Introduction

The richness of different syntactic expressions as in, e.g., first-order logic, is desirable for expressing things from different points of view or for meeting linguistic preferences of a user. On the other hand, for comparing the semantics of syntactic descriptions or for automatically processing them, normal forms may provide considerable advantages (Robinson 1965; Robinson and Voronkov 2001). In this paper, we investigate representations of knowledge bases consisting of qualitative conditionals which play a predominant role in knowledge representation and reasoning (e.g. (Adams 1975; Kraus, Lehmann, and Magidor 1990; Dubois and Prade 1994; Goldszmidt and Pearl 1996; Benferhat, Dubois, and Prade 1999; Kern-Isberner 2001)). We develop a full map of all consistent conditional knowledge bases over a small signature in different normal forms. The normal forms take various syntactical and semantical aspects into account, ranging from classical canonical disjunctive normal forms (CDNF) in the underlying propositional language to the full system P inference relation induced by a knowledge base.

Besides CDNF, we consider the antecedent normal form (ANF) (Beierle and Kutsch 2019a) and the reduced antecedent normal form (RANF) (Beierle and Haldimann 2020a), as well as their variants that also take renamings of the underlying signature into account (ρANF, ρRANF). Furthermore, we introduce new normal forms for conditional knowledge bases that take their induced system P (Lehmann and Magidor 1992) inference relation into account, the *system P normal form* (SPNF) and its renaming counterpart (ρSPNF). Focussing on the signature $\Sigma_{ab} = \{a, b\}$, our investigations show, for instance, that every of the at least 555.135.087 different consistent Σ_{ab}-knowledge bases $\mathcal{R}$ in CDNF can be transformed into a unique $\mathcal{R}'$ where $\mathcal{R}'$ is one of 262 Σ_{ab}-knowledge bases in ρSPNF and $\mathcal{R}$ and $\mathcal{R}'$ are inferentially renaming equivalent with respect to system P. In summary, the main contributions of this paper are:

- Presentation of a sequence of normal forms for conditional knowledge bases with respect to different syntactic and semantic equivalences.
- Introduction of the new normal forms SPNF and ρSPNF.
- Systematic generation and comparison of all consistent knowledge bases over Σ_{ab} in ANF, RANF, SPNF and their renaming counterparts.
- Generation of all complete system P inference relations induced by conditional knowledge bases over Σ_{ab}.

2 Background: Conditional Logic

Let $\mathcal{L}(\Sigma)$ be the propositional language over a finite signature Σ. We call a signature Σ with a linear ordering $\lessdot$ an *ordered signature* and denote it by $(\Sigma, \lessdot)$. The language may be denoted by $\mathcal{L}$ if the signature is clear from context. The formulas of $\mathcal{L}$ will be denoted by letters $A, B, C, \ldots$. We write AB for $A \wedge B$ and $\overline{A}$ for $\neg A$. We identify the set of all complete conjunctions over Σ with the set Ω of possible worlds over $\mathcal{L}$. For $\omega \in \Omega$ and $A \in \mathcal{L}$, $\omega \models A$ means that A holds in ω. The set of worlds satisfying A is $\Omega_A = \{\omega \mid \omega \models A\}$. Two formulas A, B are *equivalent*, denoted as $A \equiv B$, if $\Omega_A = \Omega_B$.

By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L})_\Sigma = \{(B|A) \mid A, B \in \mathcal{L}(\Sigma)\}$ of *conditionals* over $\mathcal{L}(\Sigma)$. Again, Σ may be omitted. As semantics for conditionals, we use *ordinal conditional functions (OCF)*, also called ranking functions, first introduced (in a more general form) in (Spohn 1988). An OCF is a function $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ expressing degrees of plausibility of possible worlds where a lower degree denotes "less surprising". At least one world must be regarded as being normal; therefore, $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$. Each κ uniquely extends to a function mapping formulas to $\mathbb{N} \cup \{\infty\}$ given by $\kappa(A) = \min\{\kappa(\omega) \mid$

$\omega \models A\}$ where $\min \emptyset = \infty$. An OCF κ *accepts* a conditional $(B|A)$, written $\kappa \models (B|A)$, if the verification of the conditional is less surprising than its falsification, i.e., if $\kappa(AB) < \kappa(A\overline{B})$; equivalently, $\kappa \models (B|A)$ iff for every $\omega' \in \Omega_{A\overline{B}}$ there is $\omega \in \Omega_{AB}$ with $\kappa(\omega) < \kappa(\omega')$. A conditional $(B|A)$ is trivial if it is *self-fulfilling* ($A \models B$) or *contradictory* ($A \models \overline{B}$); for avoiding cumbersome and non-informative case distinctions, we will restrict our attention to non-trivial conditionals. A finite, non-empty set $\mathcal{R} \subseteq (\mathcal{L}|\mathcal{L})$ of non-trivial conditionals is called a *knowledge base*. An OCF κ accepts a knowledge base $\mathcal{R}$ if κ accepts all conditionals in $\mathcal{R}$, and $\mathcal{R}$ is *consistent* if an OCF accepting $\mathcal{R}$ exists. We use $\diamond$ to denote an inconsistent knowledge base. $Mod(\mathcal{R})$ denotes the set of all OCFs κ accepting $\mathcal{R}$. Two knowledge bases $\mathcal{R}, \mathcal{R}'$ are *model equivalent*, denoted by $\mathcal{R} \equiv_{mod} \mathcal{R}'$, if $Mod(\mathcal{R}) = Mod(\mathcal{R}')$. Correspondingly, we say that $(B|A)$ and $(B'|A')$ are *equivalent*, denoted by $(B|A) \equiv (B'|A')$, if $A \equiv A'$ and $AB \equiv A'B'$.

System P (Lehmann and Magidor 1992) allows reasoning about conditional knowledge bases and consists of the following six axioms:

(RE)	for all $A \in \mathcal{L}$ it holds that $A \mid\!\sim A$
(LLE)	$A \equiv B$ and $A \mid\!\sim C$ imply $B \mid\!\sim C$
(RW)	$B \models C$ and $A \mid\!\sim B$ imply $A \mid\!\sim C$
(AND)	$A \mid\!\sim B$ and $A \mid\!\sim C$ imply $A \mid\!\sim B \wedge C$
(OR)	$A \mid\!\sim C$ and $B \mid\!\sim C$ imply $(A \vee B) \mid\!\sim C$
(CM)	$A \mid\!\sim B$ and $A \mid\!\sim C$ imply $A \wedge B \mid\!\sim C$

If B can be derived from A using the knowledge base $\mathcal{R}$ by applying the rules in system P, we denote this by $A \mid\!\sim^{p}_{\mathcal{R}} B$. It has been shown (see (Adams 1975; Pearl 1988; Dubois and Prade 1994; Lehmann and Magidor 1992)) that system P inference coincides with p-entailment (Goldszmidt and Pearl 1996) where A p-entails B in the context of $\mathcal{R}$, denoted as $A \mid\!\sim_{\mathcal{R}} B$, iff all models of $\mathcal{R}$ accept $(B|A)$.

3 Normal Form Conditionals and Renamings

For developing a method for the systematic generation of knowledge bases over a given signature Σ, we first observe that the set of syntactically different conditionals and also the set of different knowledge bases over Σ is infinite because $\mathcal{L} = \mathcal{L}(\Sigma)$ is infinite. In order to obtain a finite set, we abstract from the rich syntactic variants of the underlying propositional language $\mathcal{L}$ and represent each formula $A \in \mathcal{L}$ uniquely by its set Ω_A of satisfying worlds, called *canonical disjunctive normal form (CDNF)* of A.

Example 1. Let $\mathcal{R}_{81} = \{(\overline{a}|b), (b|a \vee b)\}$. Using the CDNF for propositional formulas, we obtain $\mathcal{R}'_{81} = \{(\{\overline{a}b, \overline{a}\overline{b}\}|\{ab, \overline{a}b\}), (\{ab, \overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\})\}$.

Example 2. Using CDNF for $\mathcal{R}_{935} = \{(\overline{a}|b), (b|a \vee b), (\overline{a} \vee b |a \vee \overline{b})\}$ we obtain $\mathcal{R}'_{935} = \{(\{\overline{a}b, \overline{a}\overline{b}\}|\{ab, \overline{a}b\}), (\{ab, \overline{a}b\} |\{ab, a\overline{b}, \overline{a}b\}), (\{ab, \overline{a}b, \overline{a}\overline{b}\}|\{ab, a\overline{b}, \overline{a}\overline{b}\})\}$.

We can further simplify the CDNF of conditionals by using *normal form conditionals* (Beierle and Kutsch 2019b). In the following proposition, the two conditions $B \subsetneq A$ and $B \neq \emptyset$ ensure the falsifiability and the verifiability of a conditional $(B|A)$, thereby excluding any trivial conditional.

Proposition 1 ($NFC(\Sigma)$ (Beierle and Kutsch 2019b)). *For $NFC(\Sigma) = \{(B|A) \mid A \subseteq \Omega_\Sigma, B \subsetneq A, B \neq \emptyset\}$, the set of* normal form conditionals *over Σ, the following holds:*

(nontrivial) *$NFC(\Sigma)$ does not contain any trivial conditional.*

(complete) *For every nontrivial conditional over Σ there is an equivalent conditional in $NFC(\Sigma)$.*

(minimal) *All conditionals in $NFC(\Sigma)$ are pairwise non-equivalent.*

Example 3. Replacing every conditional in $\mathcal{R}'_{935}$ by its equivalent normal form conditional yields $\mathcal{R}''_{935} = \{(\{\overline{a}b\} |\{ab, \overline{a}b\}), (\{ab, \overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\}), (\{ab, \overline{a}\overline{b}\}|\{ab, a\overline{b}, \overline{a}\overline{b}\})\}$.

Normal form conditionals are sufficient to represent every knowledge base up to equivalence induced by system P.

Definition 1. Two knowledge bases $\mathcal{R}, \mathcal{R}'$ are *inferentially equivalent (with respect to system P)*, denoted by $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$, if $A \mid\!\sim^{p}_{\mathcal{R}} B$ holds if and only if $A \mid\!\sim^{p}_{\mathcal{R}'} B$ for all formulas A, B.

The following is well-known (Goldszmidt and Pearl 1996):

Proposition 2. *Let $\mathcal{R}$, $\mathcal{R}'$ be knowledge bases. Then*

$$\mathcal{R} \equiv_{mod} \mathcal{R}' \text{ if and only if } \mathcal{R} \overset{p}{\sim} \mathcal{R}' \tag{1}$$

Proposition 3. *Let $\mathcal{R}, \mathcal{R}'$ be knowledge bases such that $\mathcal{R}$ is consistent and $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing every conditional $(B|A)$ in $\mathcal{R}$ by $(B'|A')$ where A' is the CDNF of A and B' is the CDNF of AB. Then $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$.*

Proof. Model equivalence of $(B|A)$ with $(AB|A)$ and its CDNF ensures $\mathcal{R} \equiv_{mod} \mathcal{R}'$, thus $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$ due to (1). $\square$

Apart from avoiding generating syntactic variants of conditionals and of knowledge bases, we also want to take symmetries into account that are induced by isomorphisms on the underlying signature. For a signature Σ, a function $\rho : \Sigma \to \Sigma'$ is a *renaming* if ρ is a bijection. For instance, the function ρ_{ab} with $\rho_{ab}(a) = b$ and $\rho_{ab}(b) = a$ is a renaming for Σ_{ab}. As usual, ρ is extended canonically to worlds, formulas, conditionals, knowledge bases, and to sets thereof.

Definition 2 ($\simeq$). Let X, X' be two signatures, worlds, formulas, knowledge bases, sets, or relations over one of these items. We say that X and X' are *isomorphic with respect to signature renamings*, denoted by $X \simeq X'$, if there exists a renaming ρ such that $\rho(X) = X'$.

For a set M, $m \in M$, and an equivalence relation $\equiv$ on M, the set of equivalence classes induced by $\equiv$ is denoted by $[M]_{/\equiv}$, and the unique equivalence class containing m is denoted by $[m]_{\equiv}$. E.g., $[\Omega_{\Sigma_{ab}}]_{/\simeq} = \{[ab], [a\overline{b}, \overline{a}b], [\overline{a}\overline{b}]\}$ are the three equivalence classes of worlds over Σ_{ab}, and we have $[(ab|ab \vee a\overline{b})]_{\simeq} = [(ab|ab \vee \overline{a}b)]_{\simeq}$.

Given a signature Σ with linear ordering $<$, in (Beierle and Haldimann 2020a) an induced linear ordering $\prec$ on $NFC(\Sigma)$ is defined. While the details of this ordering are not needed here, for an illustration, Table 1 shows some of the conditionals in $NFC(\Sigma_{ab})$ and their induced ordering $\prec$. The set $NFC(\Sigma_{ab})$ contains 50 conditionals, and $[NFC(\Sigma_{ab})]_{/\simeq}$ has 31 equivalence classes; 19 of these

class	first conditional	second conditional
[01]	r_{01}: $(\{ab\}\|\{ab, a\overline{b}\})$	r_{02}: $(\{ab\}\|\{ab, \overline{a}b\})$
[02]	r_{03}: $(\{a\overline{b}\}\|\{ab, a\overline{b}\})$	r_{04}: $(\{\overline{a}b\}\|\{ab, \overline{a}b\})$
[03]	r_{05}: $(\{ab\}\|\{ab, \overline{a}\overline{b}\})$	
[04]	r_{06}: $(\{\overline{a}\overline{b}\}\|\{ab, \overline{a}\overline{b}\})$	
[05]	r_{07}: $(\{a\overline{b}\}\|\{a\overline{b}, \overline{a}b\})$	r_{08}: $(\{\overline{a}b\}\|\{a\overline{b}, \overline{a}b\})$
...	...	...

Table 1: The first eight of the conditionals $r_{01} \prec \ldots \prec r_{50}$ in $NFC(\Sigma_{ab})$ given in CDNF for $\Sigma_{ab} = \{a, b\}$, and their equivalence classes $[01], \ldots, [31]$ induced by signature renamings

classes contain two conditionals, while the other 12 classes are singletons. The $\prec$-minimal conditional in each equivalence class is the canonical representative of that class, called *canonical normal form conditional*.

Observation 1 ($NFC(\Sigma_{ab})$)**.** The algorithm $GenKB$ (Beierle and Kutsch 2019b, Algorithm 1) generates systematically consistent knowledge bases over a given signature. It also takes renamings into account to the extent that the $\prec$-least conditional in each generated knowledge base is a canonical normal form conditional. Our implementation of $GenKB$ reveals that there are 555.135.087 such knowledge bases over Σ_{ab}. Each knowledge base contains between 1 and 25 conditionals. The number of knowledge bases of each size varies vastly. For example, $GenKB$generates only 24 knowledge bases with 25 conditionals but 88.986.856 knowledge bases with 13 conditionals.

Observation 2 ($NFC(\Sigma_{ab})$)**.** We modified the algorithm $GenKB$ in such a way that more renamings are taken into account. Instead of 555.135.087 knowledge bases, the implementation of this refined algorithm generates 364.304.482 Σ_{ab}-knowledge bases, still capturing all consistent knowledge bases over Σ_{ab} up to inferential equivalence and renaming. Since the results differ from the knowledge bases generated by $GenKB$ only by not generating some renamings, the maximal number of conditionals in a knowledge base stays the same, i.e., 25 conditionals.

4 Antecedent Normal Form

The idea of the notion of antecedentwise equivalence is to take into account the set of conditionals having the same (or propositionally equivalent) antecedent when comparing to knowledge bases.

Definition 3 ($Ant(\mathcal{R})$, $\mathcal{R}_{|A}$, ANF (Beierle and Kutsch 2019a))**.** Let $\mathcal{R}$ be a knowledge base.

- $Ant(\mathcal{R}) = \{A \mid (B|A) \in \mathcal{R}\}$ are the *antecedents of* $\mathcal{R}$.
- For $A \in Ant(\mathcal{R})$, the set $\mathcal{R}_{|A} = \{(B'|A') \mid (B'|A') \in \mathcal{R}$ and $A \equiv A'\}$ is the set of A*-conditionals in* $\mathcal{R}$.
- $\mathcal{R}$ is in *antecedent normal form (ANF)* if either $\mathcal{R}$ is inconsistent and $\mathcal{R} = \diamond$, or $\mathcal{R}$ is consistent, does not contain any self-fulfilling conditional, contains only conditionals of the form $(AB|A)$, and $\left|\mathcal{R}_{|A}\right| = 1$ for all $A \in Ant(\mathcal{R})$.

Definition 4 ($\ll_{ae}$, equivalence $\equiv_{ae}$ (Beierle and Kutsch 2019a))**.** Let $\mathcal{R}$, $\mathcal{R}'$ be knowledge bases.

- $\mathcal{R}$ is an antecedentwise equivalent sub-knowledge base of $\mathcal{R}'$, denoted by $\mathcal{R} \ll_{ae} \mathcal{R}'$, if for every $A \in Ant(\mathcal{R})$ such that $\mathcal{R}_{|A}$ is not self-fulfilling there is an $A' \in Ant(\mathcal{R}')$ with $\mathcal{R}_{|A} \equiv_{mod} \mathcal{R}'_{|A'}$.
- $\mathcal{R}$ and $\mathcal{R}'$ are *strictly antecedentwise equivalent* if $\mathcal{R} \ll_{ae} \mathcal{R}'$ and $\mathcal{R}' \ll_{ae} \mathcal{R}$.
- $\mathcal{R}$ and $\mathcal{R}'$ are *antecedentwise equivalent*, denoted by $\mathcal{R} \equiv_{ae} \mathcal{R}'$, if either both are inconsistent, or both are consistent and strictly antecedentwise equivalent.

Note that any two inconsistent knowledge bases are also antecedentwise equivalent according to Definition 4, e.g., $\{(b|a), (\overline{b}|b)\} \equiv_{ae} \{(b|b), (a\overline{a}|\top)\}$, enabling us to avoid cumbersome case distinctions when dealing with consistent and inconsistent knowledge bases.

Example 4. The two knowledge bases $\{(a|a \vee b), (b|a \vee b)\}$ and $\{(ab|a \vee b)\}$ are antecedentwise equivalent. $\mathcal{R}''_{935}$ from Example 3 is in ANF.

Antecedentwise equivalence ensures inferential equivalence.

Proposition 4. *If* $\mathcal{R}, \mathcal{R}'$ *are knowledge bases,* $\mathcal{R} \equiv_{ae} \mathcal{R}'$ *implies* $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$.

Proof. W.l.o.g. assume that $\mathcal{R}$ and $\mathcal{R}'$ are consistent. From $\mathcal{R} \equiv_{ae} \mathcal{R}'$, we get $\mathcal{R} \ll_{ae} \mathcal{R}'$, and Definition 4 implies that there is a function $f : Ant(\mathcal{R}) \to Ant(\mathcal{R}')$ with $\mathcal{R}_{|A} \equiv_{mod} \mathcal{R}'_{|f(A)}$ for each $A \in Ant(\mathcal{R})$. Thus, $\mathcal{R} = \bigcup_{A \in Ant(\mathcal{R})} \mathcal{R}_{|A} \equiv_{mod} \bigcup_{A \in Ant(\mathcal{R})} \mathcal{R}'_{|f(A)} \subseteq \mathcal{R}'$ implies $Mod(\mathcal{R}') \subseteq Mod(\mathcal{R})$. Employing this argumentation in both directions, we get $\mathcal{R} \equiv_{mod} \mathcal{R}'$ and therefore also $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$ due to Proposition 2. $\square$

A set of transformation rules can map every knowledge base into its uniquely determined ANF (Beierle and Kutsch 2019a). Furthermore, the algorithm KB^{ae}_{gen} (Beierle and Kutsch 2019a, Algorithm 1) generates systematically consistent knowledge bases over a given signature in ANF using only normal form conditionals; it takes renaming into account as sketched in Observation 1 for the algorithm $GenKB$, i.e., the $\prec$-least conditional in each generated knowledge base is a canonical normal form conditional.

Observation 3 (ANF)**.** Our implementation of KB^{ae}_{gen} shows that there are 758.808 knowledge bases over Σ_{ab} in ANF such that the $\prec$-least conditional in the knowledge base is a canonical normal form conditional. Note that compared to the large numbers of 555.135.087 and 364.304.482 Σ_{ab}-knowledge bases generated by $GenKB$ (cf. Observations 1 and 2), this huge reduction is achieved solely by employing the ANF of knowledge bases. As the ANF merges conditionals with the same antecedent, the knowledge bases also tend to be smaller. The 1.827 largest knowledge bases generated by KB^{ae}_{gen} contain 11 conditionals.

The following proposition shows that KB^{ae}_{gen} captures all consistent knowledge bases over a given signature up to inferential equivalence and renamings.

Proposition 5 (KB^{ae}_{gen})**.** *Applying KB^{ae}_{gen} to an ordered signature $(\Sigma, \lessdot)$ terminates and returns a set of knowledge bases $\mathcal{KB}$ for which the following holds:*

($\overset{p}{\sim}$ completeness) *If $\mathcal{R}$ is a consistent Σ-knowledge base then there is $\mathcal{R}' \in \mathcal{KB}$ and a signature renaming ρ such that $\mathcal{R} \overset{p}{\sim} \rho(\mathcal{R}')$.*

Proof. For every consistent Σ-knowledge base $\mathcal{R}$ there is $\mathcal{R}' \in \mathcal{KB}$ and a signature renaming ρ such that $\mathcal{R} \equiv_{mod} \rho(\mathcal{R}')$ (Beierle and Kutsch 2019a, Proposition 7), and thus $\mathcal{R} \overset{p}{\sim} \rho(\mathcal{R}')$ due to (1). □

Observation 4 (ANF)**.** The 758.808 knowledge bases over Σ_{ab} generated by KB^{ae}_{gen} capture all consistent Σ_{ab}-knowledge bases up to renaming and inferential equivalence with respect to system P inference.

5 Knowledge Bases in RANF and in ρRANF

While the antecedent normal form can compact conditional knowledge bases drastically, a knowledge base in ANF may still contain redundancies in form of conditionals that can be removed without changing the system P inference relation induced by the knowledge base.

Example 5. Consider $\mathcal{R} = \{(ab|a), (ab|b), (ab|a \vee b)\}$. The third conditional can be derived from the first two conditionals with system P axiom (OR).

Proposition 6. *Let $\mathcal{R}$ be a knowledge base and $A, B \in \mathcal{L}$ such that $A \mathrel{|\!\sim}^{p}_{\mathcal{R}} B$. Then $\mathrel{|\!\sim}^{p}_{\mathcal{R}} = \mathrel{|\!\sim}^{p}_{\mathcal{R} \cup \{(B|A)\}}$.*

Proof. Let $(D|C) \in (\mathcal{L} \mid \mathcal{L})$ be a conditional. If $(D|C)$ has a system P derivation from $\mathcal{R}$, then it has a system P derivation from $\mathcal{R} \cup \{(B|A)\}$ because of the semi-monotonicity of system P. If $(D|C)$ can be derived from $\mathcal{R} \cup \{(B|A)\}$ with system P, then it can be derived from $\mathcal{R}$ as well because $(B|A)$ can be derived from $\mathcal{R}$. □

In Example 5, this implies that omitting the conditional $(ab|a \vee b)$ does not change the system P inference relation induced by the knowledge base. The *reduced form* is a knowledge base avoiding such redundancies.

Definition 5 (reduced form, RANF (Beierle and Haldimann 2020a))**.** Let $\mathcal{R}$ be a knowledge base.

- $\mathcal{R}$ is in *reduced form* (with respect to system P) if there is no conditional $(B|A) \in \mathcal{R}$ such that $A \mathrel{|\!\sim}^{p}_{\mathcal{R} \setminus (B|A)} B$.
- $\mathcal{R}$ is in *reduced antecedent normal form (RANF)* if $\mathcal{R}$ is in ANF and in reduced form.

In (Beierle and Haldimann 2020a), a transformation system Θ^{ra} is provided such that every $\mathcal{R}' \in \Theta^{ra}(\mathcal{R})$ is in RANF and model equivalent to $\mathcal{R}$.

The approaches to generate knowledge bases examined in Sec. 3 and 4 consider renamings to some extent, but the mentioned algorithms still generate some knowledge bases that are isomorphic with respect to signature renaming to another generated knowledge base. Also, the normal forms CDNF, NFC, ANF, and RANF do not take renamings into account. The *renaming normal form* avoids redundancies regarding renamings because any two knowledge bases in renaming normal form that are not equal are not isomorphic with respect to signature renaming. To define the renaming normal form, we will rely on the linear ordering $\prec$ on $NFC(\Sigma)$ (Beierle and Haldimann 2020a) sketched in Section 3, cf. Table 1. We extend $\prec$ to an ordering on knowledge bases.

Definition 6 ($\preccurlyeq_{set}$)**.** The lexicographic extension of the ordering $\preccurlyeq$ on $NFC(\Sigma)$ to strings over $NFC(\Sigma)$ is denoted by $\preccurlyeq_{lex}$. For knowledge bases $\mathcal{R} = \{r_1, \ldots, r_n\}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_{n'}\}$ over $NFC(\Sigma)$ with $r_i \prec r_{i+1}$ and $r'_j \prec r'_{j+1}$ the ordering $\preccurlyeq_{set}$ is given by: $\mathcal{R} \preccurlyeq_{set} \mathcal{R}'$ iff $n < n'$, or $n = n'$ and $r_1 \ldots r_n \preccurlyeq_{lex} r'_1 \ldots r'_{n'}$

Proposition 7. *The ordering $\preccurlyeq_{set}$ is a linear ordering on the set of knowledge bases over $NFC(\Sigma)$.*

In the following, we will abbreviate $\mathcal{R} \prec_{set} \mathcal{R}'$ simply by $\mathcal{R} \prec \mathcal{R}'$, and analogously for the non-strict version $\preccurlyeq_{set}$.

Definition 7 (ρNF (Beierle and Haldimann 2020a))**.** A knowledge base $\mathcal{R} \subseteq NFC(\Sigma)$ is in *renaming normal form* (ρNF) if for every $\mathcal{R}'$ with $\mathcal{R} \simeq \mathcal{R}'$ it holds that $\mathcal{R} \preccurlyeq \mathcal{R}'$.

Note that we defined the ρNF only for knowledge bases containing normal form conditionals; a generalized definition of the ρNF for arbitrary conditionals is given in (Beierle and Haldimann 2020b). The ρNF can be combined with other normal forms. A knowledge base is in *renaming reduced antecedent normal form* (ρRANF) if it is both in RANF and in ρNF, and it is in *renaming antecedent normal form* (ρANF) if it is both in ρNF and in ANF. A list of knowledge bases in ρRANF can be generated by algorithm $KB^{\rho ra}$ (Beierle and Haldimann 2020a, Algorithm 2).

Proposition 8 ($KB^{\rho ra}$)**.** *Applying $KB^{\rho ra}$ to an ordered signature $(\Sigma, \lessdot)$ terminates and returns a set of knowledge bases $\mathcal{KB}$ such that:*

($\overset{p}{\sim}$ completeness) *If $\mathcal{R}$ is a consistent Σ-knowledge base then there is $\mathcal{R}' \in \mathcal{KB}$ and a signature renaming ρ such that $\mathcal{R} \overset{p}{\sim} \rho(\mathcal{R}')$.*

Proof. The proof follows from model completeness of $KB^{\rho ra}$ (Beierle and Haldimann 2020a, Prop. 11) and Proposition 2. □

Observation 5 (ρRANF)**.** $KB^{\rho ra}$ generates systematically consistent knowledge bases in ρRANF over a given signature using only normal form conditionals. Our implementation of $KB^{\rho ra}$ shows that there are exactly 2.143 Σ_{ab}-knowledge bases, with at most 4 conditionals, in ρRANF.

From the list of knowledge bases in ρRANF we can easily obtain the set of all knowledge bases in RANF. Algorithm KB^{ra}(Algorithm 1) shows one way to do this.

Proposition 9 (KB^{ra})**.** *Algorithm 1 generates all knowledge bases in RANF, i.e., applying KB^{ra} to an ordered signature $(\Sigma, \lessdot)$ terminates and returns a set of knowledge bases $\mathcal{KB}$ such that:*

(RANF) *All knowledge bases in $\mathcal{KB}$ are in RANF.*

(RANF completeness) *If a consistent knowledge base $\mathcal{R} \in NFC(\Sigma)$ is in RANF, then $\mathcal{R} \in \mathcal{KB}$.*

($\overset{p}{\sim}$ completeness) *If $\mathcal{R}$ is a consistent Σ-knowledge base then there is $\mathcal{R}' \in \mathcal{KB}$ such that $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$.*

Algorithm 1 KB^{ra} – Algorithm to generate all knowledge bases in RANF. Perm_Σ denotes the set of all signature renamings in Σ including the identity.

Require: Ordered signature $(\Sigma, \lessdot)$
Ensure: Set of knowledge bases in RANF over Σ
$\mathcal{KB}_{\rho RANF} \leftarrow KB^{\rho ra}(\Sigma)$
$\mathcal{KB}_{RANF} \leftarrow \emptyset$
for all $\rho \in \mathrm{Perm}_\Sigma$ **do**
  $\mathcal{KB}_{RANF} \leftarrow \mathcal{KB}_{RANF} \cup \rho(\mathcal{KB}_{\rho RANF})$
end for
return $\mathcal{KB}_{RANF}$

Proof. (RANF) All knowledge bases in $\mathcal{KB}_{\rho RANF} = KB^{\rho ra}(\Sigma)$ are in RANF. Applying a renaming preserves the RANF. Therefore, all knowledge bases in $\mathcal{KB}$ are in RANF.

(RANF completeness) Let $\mathcal{R}$ be a knowledge base in RANF. Then there is a $\mathcal{R}'$ in ρRANF and a renaming ρ such that $\mathcal{R}' = \rho(\mathcal{R})$. Proposition 8 states that $\mathcal{R} \in KB^{\rho ra}(\Sigma)$ because $\mathcal{R}$ is in ρRANF. As all renamings, including the inverse of ρ, are applied to $\mathcal{R}'$, $\mathcal{R}$ is added to $\mathcal{KB}$.

($\overset{p}{\sim}$ completeness) For every knowledge base $\mathcal{R}$ there is a $\mathcal{R}'$ in RANF that is inferentially equivalent to $\mathcal{R}$. With (RANF completeness) it follows that $\mathcal{R}' \in \mathcal{KB}$. ☐

Observation 6 (RANF)**.** KB^{ra} generates systematically consistent knowledge bases over a given signature in RANF using only normal form conditionals. Our implementation of KB^{ra} shows that there are exactly 4.168 knowledge bases in RANF over Σ_{ab}, containing at most 4 conditionals.

6 System P Inference Relations over Σ_{ab}-Knowledge Bases

In the previous sections, we introduced different normal forms for conditional knowledge bases all of which respect inferential equivalence with respect to system P. There are still different knowledge bases in RANF (or ρRANF) that are inferentially equivalent with respect to system P inference. In this section, we want to take this equivalence into account.

Proposition 10. *For every system P inference relation* $\mid\!\sim$ *there is a conditional knowledge base* $\mathcal{R}$ *in RANF such that* $\mid\!\sim \; = \; \mid\!\sim^p_{\mathcal{R}}$.

Proof. Let $\mid\!\sim$ be a system P inference relation. The set $\mathcal{R}' = \{(B|A) \in NFC(\Sigma) \mid A \mid\!\sim B\}$ is a finite knowledge base. $\mathcal{R}'$ is complete in the sense that we cannot derive any normal form conditional from $\mathcal{R}'$ with system P that is not already in $\mathcal{R}'$. Consider any conditional $(C|B)$. Let $(C'|B')$ be the normal form conditional that is equivalent to $(C|B)$. All of the four statements $B \mid\!\sim C$, $B' \mid\!\sim C'$, $(C'|B') \in \mathcal{R}'$, and $B' \mid\!\sim^p_{\mathcal{R}'} C'$ are equivalent to each other. Therefore, $\mid\!\sim \; = \; \mid\!\sim^p_{\mathcal{R}'}$.

There exists a knowledge base $\mathcal{R}$ in RANF, that is model equivalent to $\mathcal{R}'$ (Beierle and Haldimann 2020a, Proposition 2). Because of Proposition 2, $\mathcal{R}$ and $\mathcal{R}'$ are inferentially equivalent and therefore $\mid\!\sim \; = \; \mid\!\sim^p_{\mathcal{R}'} = \; \mid\!\sim^p_{\mathcal{R}}$. ☐

Based on the equivalence relation $\overset{p}{\sim}$ and the ordering $\preccurlyeq$ on knowledge bases, we can define a unique normal form for each equivalence class with respect to $\overset{p}{\sim}$.

Definition 8 (SPNF)**.** A knowledge base $\mathcal{R}$ is in *system P normal form* (SPNF) if $\mathcal{R}$ is in RANF and for every knowledge base $\mathcal{R}' \subseteq NFC(\Sigma)$ in RANF with $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$ it holds that $\mathcal{R} \preccurlyeq \mathcal{R}'$.

As there are 256 different conditionals in CDNF (including trivial conditionals), we store a list of 256 Booleans for every inference relation. This representation is suitable for every inference relation, that does not distinguish syntactic variants of the formulas in the query. To generate the desired list of inference relations, we begin with the set of all knowledge bases in RANF, as generated by KB^{ra}. Then we calculate the inference relation for each of these knowledge bases. After removing duplicates, this is the list of all inference relations.

Observation 7. Our implementation shows that there are exactly 484 different system P inference relations over Σ_{ab}, and the same number of Σ_{ab}-knowledge bases in SPNF.

Proposition 11. *If a knowledge base* $\mathcal{R}$ *is in SPNF, then for every other consistent knowledge base* $\mathcal{R}' \subseteq NFC(\Sigma)$ *with* $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$ *it holds that* $\mathcal{R} \preccurlyeq \mathcal{R}'$.

Proof. Let $\mathcal{R}$ be in SPNF. Assume, there is a consistent $\mathcal{R}' \subseteq NFC(\Sigma)$ such that $\mathcal{R}' \overset{p}{\sim} \mathcal{R}$ and $\mathcal{R}' \prec \mathcal{R}$. Let $\mathcal{R}'' \in \Theta^{ra}(\mathcal{R}')$ be a knowledge base in RANF that is equivalent to $\mathcal{R}'$. Because Θ^{ra} can only reduce the number of conditionals in $\mathcal{R}'$ (the knowledge base is consistent and contains only normal form conditionals), we have $\mathcal{R}'' \preccurlyeq \mathcal{R}' \prec \mathcal{R}$. Furthermore, $\mathcal{R}''$ is model equivalent and therefore inferentially equivalent (with respect to system P) to $\mathcal{R}'$. This is a contradiction to $\mathcal{R}$ being in SPNF. ☐

Proposition 12. *For every consistent* $\mathcal{R} \subseteq NFC(\Sigma)$ *there is a knowledge base* $\mathcal{R}'$ *in SPNF with* $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$.

Proof. Let $\mathcal{R} \subseteq NFC(\Sigma)$. Let $\mathcal{R}''$ be a knowledge base in RANF that is equivalent to $\mathcal{R}$. Therefore, the equivalence class $[\mathcal{R}'']_{\overset{p}{\sim}}$ is non-empty. Because there are only finitely many knowledge bases in RANF, there is a minimum $\mathcal{R}'$ in $[\mathcal{R}'']_{\overset{p}{\sim}}$. $\mathcal{R}'$ is in SPNF and $\mathcal{R} \overset{p}{\sim} \mathcal{R}'$. ☐

Now we also want to take renamings into account.

Definition 9 ($\overset{p}{\simeq}$)**.** Two knowledge bases $\mathcal{R}, \mathcal{R}'$ are *inferentially equivalent up to signature renamings with respect to system P inference*, denoted as $\mathcal{R} \overset{p}{\simeq} \mathcal{R}'$, if there is a renaming ρ such that $\rho(\mid\!\sim^p_{\mathcal{R}}) = \; \mid\!\sim^p_{\mathcal{R}'}$.

Definition 10 (ρSPNF)**.** A knowledge base $\mathcal{R}$ is in *renaming system P normal form* (ρSPNF) if $\mathcal{R}$ is in RANF and for every other knowledge base $\mathcal{R}' \subseteq NFC(\Sigma)$ in RANF with $\mathcal{R} \overset{p}{\simeq} \mathcal{R}'$ it holds that $\mathcal{R} \preccurlyeq \mathcal{R}'$.

Proposition 13. *If a knowledge base* $\mathcal{R}$ *is in* ρ*SPNF, then for every other knowledge base* $\mathcal{R}' \subseteq NFC(\Sigma)$ *with* $\mathcal{R} \overset{p}{\simeq} \mathcal{R}'$ *it holds that* $\mathcal{R} \preccurlyeq \mathcal{R}'$.

Proof. Let $\mathcal{R}$ be in ρSPNF. Assume there is a consistent $\mathcal{R}' \subseteq NFC(\Sigma)$ such that $\mathcal{R}' \overset{p}{\simeq} \mathcal{R}$ and $\mathcal{R}' \prec \mathcal{R}$. Let $\mathcal{R}'' \in \Theta^{ra}(\mathcal{R}')$ be a knowledge base in RANF that is equivalent to $\mathcal{R}'$. Because Θ^{ra} can only reduce the number of conditionals in $\mathcal{R}'$ (the knowledge base is consistent and contains only normal form conditionals), we have $\mathcal{R}'' \preccurlyeq \mathcal{R}' \prec \mathcal{R}$. Furthermore, $\mathcal{R}''$ and $\mathcal{R}'$ are model equivalent and due to (1) inferentially equivalent with respect to system P. Hence, $\mathcal{R}'' \overset{p}{\simeq} \mathcal{R}'$ and $\mathcal{R}' \overset{p}{\simeq} \mathcal{R}$. This is a contradiction to $\mathcal{R}$ being in ρSPNF. □

Proposition 14. *For every system P inference relation $\mid\!\sim$ there is a conditional knowledge base $\mathcal{R}$ in ρSPNF and a renaming ρ such that $\mid\!\sim\, = \rho(\mid\!\sim^{p}_{\mathcal{R}})$.*

Proof. Let $\mid\!\sim$ be a system P inference relation. There is a knowledge base $\mathcal{R}'$ in RANF such that $\mid\!\sim\, = \,\mid\!\sim^{p}_{\mathcal{R}'}$ (see Proposition 10). Therefore, the equivalence class $[\mathcal{R}']_{\overset{p}{\simeq}}$ is non-empty. Because there are only finitely many knowledge bases in RANF, there is a minimum $\mathcal{R}$ in $[\mathcal{R}']_{\overset{p}{\simeq}}$. $\mathcal{R}$ is in ρSPNF and because $\mathcal{R} \overset{p}{\simeq} \mathcal{R}'$ there is a renaming ρ such that $\rho(\mid\!\sim^{p}_{\mathcal{R}}) = \,\mid\!\sim^{p}_{\mathcal{R}'} = \,\mid\!\sim$. □

Observation 8. Our implementation shows that, up to renamings of the underlying signature, there are exactly 262 different system P inference relations over Σ_{ab}, and correspondingly, 262 Σ_{ab}-knowledge bases in ρSPNF.

We can also relate knowledge bases $\mathcal{R}, \mathcal{R}'$ by their acceptance of so-called *base conditionals*, i.e., normal form conditionals $(B|A)$ with $|A| = 2$ and $|B| = 1$, inducing the relation $R \sim_{base} \mathcal{R}'$ that holds if, for all base conditionals $(B|A)$, we have $A \mid\!\sim^{p}_{\mathcal{R}} B$ iff $A \mid\!\sim^{p}_{\mathcal{R}'} B$. Obviously, $\sim_{base}$ is an equivalence relation, and $\overset{p}{\sim}$ is a refinement of $\sim_{base}$.

Observation 9. There are 219 equivalence classes with respect to $\sim_{base}$ over all knowledge bases in RANF over Σ_{ab}.

A base conditional $r = (\{\omega_1\}|\{\omega_1, \omega_2\})$ encodes that ω_1 is strictly more plausible than ω_2. This means that $\kappa(\omega_1) < \kappa(\omega_2)$ for all ranking models κ of a knowledge base containing r. Note that the 219 different equivalence classes with respect to $\sim_{base}$ coincide with the 219 different partial orders on the four worlds over Σ_{ab}.

7 Conclusions and Further Work

We introduced the normal forms SPNF and ρSPNF for conditional knowledge bases and developed a complete map of all Σ_{ab}-knowledge bases with respect to these and several other normal forms. For instance, there are only 262 Σ_{ab}-knowledge bases in ρSPNF, and each consistent Σ_{ab}-knowledge base has a unique ρSPNF. We are currently extending our investigations to signatures with more elements and by taking inference relations other than the induced system P inference relation into account. These other inference relations include system Z inference (Goldszmidt and Pearl 1996) and inference relations obtained from different inference modes with respect to various classes of models (Beierle et al. 2021).

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