System Z for Conditional Belief Bases with Positive and Negative Information

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Abstract

In non-monotonic reasoning, conditional belief bases mostly contain positive information in the form of standard conditionals. However, in practice we are often confronted with negative information, stating that a conditional does *not* hold, i.e. we need a suitable approach for reasoning over belief bases Δ with positive and negative information. In this paper, we investigate the interaction of positive and negative information in a conditional belief base and establish a property for partitions of Δ that is equivalent to consistency. Based on this property, we develop a non-trivial extension of system Z for mixed conditional belief bases and provide an algorithm to compute this partition.

1 Introduction

The idea of drawing inferences from sets of conditionals has been investigated for quite a long time, with very well-known approaches like rational closure (Lehmann and Magidor 1992) and System Z (Pearl 1990). It was formulated for sets containing positive information in the form of statements *'If A then B'*, i.e. conditionals $(B|A)$. But not only positive conditional information is relevant for non-monotonic reasoning, also negative conditional information has to be processed, i.e. negative conditional assertions claiming that a conditional $(B|A)$ does *not* hold. Note that declaring that a conditional $(B|A)$ does not hold is certainly not the same as declaring that the negation $(\neg B|A)$ holds. How deeply the relevance of negative information is enrooted in non-monotonic reasoning gets clear if we take a look on *Rational Monotonicity*, one of the properties of rational consequence relations defined by Gabbay and Makinson in (1991; 1989), which states that

(RMO) If $(B|A)$ and not$((\neg C|A))$, then $(B|A \wedge C)$.

(RMO) makes use of negative information, as to express that a statement is not true. And the fact that an agent can know, or believe, that an assertion is true surely entails that she can also know, or believe, that it is *not* true.

In this paper, we express these negative assertions using weak conditionals $(|D|C|)$, stating that *'If C, then D might be true but $\neg D$ is not plausible'*, i.e. the acceptance of D is not guaranteed if C is accepted but might be possible. And therefore, $(|D|C|)$ displays that the standard conditional $(\neg D|C)$ is not accepted without further elaborating on the negation of it. Weak conditionals as negative information about the acceptance of conditionals also play a crucial role in (Eichhorn, Kern-Isberner, and Ragni 2018; Sauerwald, Kern-Isberner, and Beierle 2020). Rott introduced in (Rott 2019) difference-making conditionals $A \gg B$ (*'B because A'*) as an extension of standard conditionals that take into account a fundamental feature of conditionals used in natural language: typically the antecedent is relevant to the consequent. This notion of relevance, specified by the so-called *Relevant Ramsey Test*, decomposes into a positive and a negative part, which can be expressed by sets of both standard and weak conditionals $A \gg B = \{(B|A), (|\overline{B}|\overline{A}|)\}$, as was shown in (Sezgin, Kern-Isberner, and Rott 2020). We illustrate belief bases with different kinds of conditionals in an example:

Example 1. *An agent takes part in a botany class and learns a lot about berries. At the end of class, the agent is a bit confused, so she decides to write down her beliefs as a set of conditionals. She learns that a fruit is a berry (b) because it has seeds (s), so far the agent assumed that if a plant has seeds then it is a vegetable (v). Moreover, the agents knows that not all vegetables have seeds, e.g. lettuce, and also that berries are not vegetables. Her beliefs are captured by the following conditional belief base* $\Delta = \{s \gg b, (v|s), (|s|v|), (\bar{v}|b)\}$. *Now, she tries to decide whether these conditionals are conflicting with each other.*

Recognizing the value of taking belief bases with both positive and negative information into consideration, Booth and Paris (Booth and Paris 1998) extended rational closure to these mixed sets of conditionals. They also provide an algorithm which constructs the rational closure and yields completeness theorems for the conditional assertions entailed by such a mixed belief base. One of the most basic prerequisites required to be able to trust these inferences is that the set of rules defining them is consistent. Booth and Paris assumed throughout their work, that the belief bases they were examining are satisfiable. For sets of standard conditionals, we know that in general it is not trivial to confirm whether such a set is consistent or not. Adams provided in (1965) a suitable definition of consistency based on the notion of tolerance, leading to an elegant way to check whether a conditional belief base is consistent or not. Based on his notion of tolerance, Pearl defined in (1990) a system equivalent to rational closure which he called *System Z*. In this paper, we close the gap between the work of Booth and Paris and the consistency conditions defined by Pearl and Adams, by providing an algorithm that decides whether a mixed set of conditionals is consistent or not. Moreover, we define an extended System Z for belief bases with both positive and negative information. As we have seen before, weak conditionals resp. negative information introduce interesting dynamics and a change of perspective to non-monotonic reasoning, therefore the question whether such a set is consistent becomes more urgent, and it is not trivial to define an extension of System Z serving these mixed sets of conditionals. Our main contributions are the following:

- We define a fixed-point operator Λ_Δ in order to explore the interplay of weak conditionals among each other when combined with standard conditionals.
- We introduce a property $(*)$ for partitions of sets containing standard and weak conditionals, whose fulfillment is equivalent to the consistency of such sets.
- We present an algorithm that determines a partition satisfying $(*)$ for sets of standard and weak conditionals, enabling us to define a Z^w-ordering of the rules.
- We introduce a nontrivial extension of System Z ranking functions for sets of conditionals with both positive and negative information.
- We prove that the existence of an output partition of the above mentioned algorithm is equivalent to the consistency of a set of standard and weak conditionals, making it an adequate consistency test for such sets.

The rest of this paper is organized as follows: In section 2 we present relevant formal preliminaries. The next section deals with the impact of weak conditionals on the consistency of mixed sets of conditionals Δ. Weak System Z^w and a consistency test for Δ is introduced in section 4. Finally, section 5 concludes.

2 Formal Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language over an alphabet Σ with atoms $a, b, c, \ldots$ and with formulas $A, B, C, \ldots$. For conciseness of notation, we will omit the logical and-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e., $\overline{A}$ means $\neg A$. The set of all propositional interpretations over Σ is denoted by Ω_Σ. As the signature will be fixed throughout the paper, we will usually omit the subscript and simply write Ω. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $Mod(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $Mod(A) \subseteq Mod(B)$, as usual. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise. The set of classical consequences of a set of formulas $\mathcal{A} \subseteq \mathcal{L}$ is $Cn(\mathcal{A}) = \{B \mid \mathcal{A} \models B\}$. The deductively closed set of formulas which has exactly a subset $\mathcal{W} \subseteq \Omega$ as a model is called the *formal theory of* $\mathcal{W}$ and defined as $Th(\mathcal{W}) = \{A \in \mathcal{L} \mid \omega \models A \text{ for all } \omega \in \mathcal{W}\}$.

We extend $\mathcal{L}$ to a conditional language $(\mathcal{L}|\mathcal{L})$ by introducing a conditional operator $(\cdot|\cdot)$, so that $(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$. $(\mathcal{L}|\mathcal{L})$ is a flat conditional language, no nesting of conditionals is allowed. A is called the antecedent of $(B|A)$, and B is its consequent. $(B|A)$ expresses *'If A, then (plausibly) B'*. According to de Finetti (1975), conditionals can be regarded as three-valued logical entities on possible worlds $\omega \in \Omega$, distinguishing between verification $\omega \models AB$, falsification $\omega \models A\overline{B}$ and neutrality $\omega \models \overline{A}$. For a conditional $(B|A)$, $(\overline{B}|A)$ is the strict negation of the conditional. In the following, conditionals $(B|A) \in (\mathcal{L}|\mathcal{L})$ are referred to as *standard conditionals* or, if there is no danger of confusion, simply *conditionals*.

We further extend our framework of conditionals to a language with *weak conditionals* $(|\mathcal{L}|\mathcal{L}|)$ by introducing a weak conditional operator $(|\cdot|\cdot|)$. For a weak conditional $(|D|C|)$, we call C the antecedent and D the consequent. As for standard conditionals, $(|\mathcal{L}|\mathcal{L}|)$ is a flat conditional language, and $(|D|C|)$ expresses *'If C, then D might be the case but $\overline{D}$ is not plausible'*. In a way, the weak conditional $(|D|C|)$ is the negation of the standard conditional $(\overline{D}|C)$ (Lewis 1973). The former is accepted iff the latter is not. The evaluation of a weak conditional corresponds to the evaluation of the standard conditional, with the same definition of verification, falsification and neutrality.

A (conditional) *belief base* is a finite set of conditionals $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \cup \{(|D_1|C_1|), \ldots, (|D_m|C_m|)\}$. We write $\Delta^s = \{(B_i|A_i)\}_{1 \leqslant i \leqslant n}$ for the set of standard conditionals, and $\Delta^w = \{(|D_l|C_l|)\}_{1 \leqslant l \leqslant m}$ for the set of weak conditionals. Throughout this paper, our belief base $\Delta = \Delta^s \cup \Delta^w$ will consist of both standard and weak conditionals. Since we examine the interplay of different types of conditionals, we write (Y/X) both for $(Y|X)$ and $(|Y|X|)$ as a placeholder for cases in which we make statements which hold for both standard and weak conditionals.

Definition 1 (Tolerance for (Y/X)). *A conditional (Y/X) is tolerated by a set of conditionals Δ, if there is a world $\omega \in \Omega$ that verifies the conditional and does not falsify any of the conditionals in Δ.*

To give an appropriate semantics to (standard resp. weak) conditionals and belief bases, we need richer semantic structures like epistemic states in the sense of Halpern (2003). In this paper, we build upon ordinal conditional functions (Spohn 1988).

Ordinal conditional functions (OCFs, also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$, with $\kappa^{-1}(0) \neq \emptyset$, assign to each world ω an implausibility rank $\kappa(\omega)$. OCFs were first introduced by Spohn (1988). The higher $\kappa(\omega)$, the less plausible ω is, and the normalization constraint requires that there are worlds having maximal plausibility. We have $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$, and in particular, $\kappa(\bot) = \infty$. Due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A)$ and $\kappa(\overline{A})$ must be 0. A proposition A is believed if $\kappa(\overline{A}) > 0$, and the belief set of a ranking function κ is defined as $Bel(\kappa) = Th(\kappa^{-1}\{0\})$.

Definition 2 (Acceptance of (Y/X) by OCFs)**.** *A (standard) conditional $(B|A)$ is accepted in an epistemic state represented by an OCF κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$ or $\kappa(A) = \infty$.*

A weak conditionals $(|D|C|)$ is accepted in an epistemic state represented by an OCF κ, written as $\kappa \models (|D|C|)$, if and only if $\kappa \not\models (\overline{D}|C)$ or $\kappa(C) = \infty$, i.e., $\kappa(CD) \leqslant \kappa(C\overline{D})$ or $\kappa(C) = \infty$.

For the acceptance of a standard conditional, the verification of $(B|A)$ must be more plausible than its falsification, or the premise of the conditional is always false. Note that accepting a weak conditional is not equivalent to the acceptance of the conditional with negated consequent ($\kappa \models (\overline{D}|C)$) but weaker since it allows for indifference between CD and $C\overline{D}$. In this case both $(D|C)$ and $(\overline{D}|C)$ fail to be accepted.

A conditional belief base Δ is *consistent* if there is an OCF κ such that $\kappa \models (Y/X)$ for all $(Y/X) \in \Delta$.

3 Limiting weak conditionals

In this section, we investigate consistency of sets of standard and weak conditionals. We first examine the impact of adding negative information resp. weak conditionals to the belief base and introduce a set of limiting weak conditionals which plays a crucial role for defining our extended System Z.

Since weak conditionals have a less strict acceptance condition and therefore impose a weaker impact on the consistency of a set Δ, we cannot simply apply the standard methods for determining the consistency of a set Δ. In principle, weak conditionals do not impose restriction of tolerance onto each other, since their acceptance condition is weakened. But if we add standard conditionals that impose a certain ordering of worlds, where worlds verifying the standard conditionals strictly have to be more plausible than worlds falsifying them, the interplay of weak conditionals changes and has to adapt to this new ordering. With every weak conditional incorporated into this ordering, we have to check the acceptance of the other weak conditionals again. This dynamic is crucial and it is not trivial to fully serve it. In order to do so, we define an operator Λ_Δ that determines the set of weak conditionals that are not tolerated by an arbitrary subset of Δ:

Definition 3 (Operator Λ_Δ)**.** *Let $\Delta = \Delta^s \cup \Delta^w$ be a set of standard and weak conditionals and $\widetilde{\Delta}^w \subseteq \Delta^w$. Then $\Lambda_\Delta(\widetilde{\Delta}^w)$ is the set of weak conditionals of Δ that are not tolerated by $\Delta^s \cup \widetilde{\Delta}^w$, i.e. $(|D|C|) \in \Lambda_\Delta(\widetilde{\Delta}^w)$ iff*

$$CD \wedge \bigwedge_{(B_i|A_i)\in\Delta^s} (A_i \Rightarrow B_i) \wedge \bigwedge_{(|D_l|C_l|)\in\Lambda_\Delta(\widetilde{\Delta}^w)} (C_l \Rightarrow D_l) \equiv \bot. \quad (1)$$

Λ_Δ is a monotonic operator, i.e. for $\widetilde{\Delta}^w_1 \subseteq \widetilde{\Delta}^w_2$, it holds that $\Lambda_\Delta(\widetilde{\Delta}^w_1) \subseteq \Lambda_\Delta(\widetilde{\Delta}^w_2)$. Furthermore, note that Λ_Δ is bounded, since $\Lambda_\Delta(\widetilde{\Delta}^w) \subseteq \Delta^w$. Therefore, we can find a least fixed point of Λ_Δ by repeated application, starting with the empty set:

$$\Lambda_\Delta \uparrow 0 := \emptyset \text{ and } \Lambda_\Delta \uparrow (k+1) := \Lambda_\Delta(\Lambda_\Delta \uparrow k) \quad (k \in \mathbb{N}).$$

In the following, the least fixed point $\Lambda_\Delta \uparrow k$ of Λ_Δ plays a crucial role:

Definition 4 (Set of limiting weak conditionals)**.** *For a set of standard and weak conditionals $\Delta = \Delta^s \cup \Delta^w$, we define the* set of limiting weak conditionals *Δ^w_* as the least fixed point of Λ_Δ.*

It is clear that Δ^w_* is unique and the following proposition holds:

Proposition 1. *Let $\Delta = \Delta^s \cup \Delta^w$ be a set of standard and weak conditionals and Δ^w_* be the corresponding set of limiting conditionals. Then all $(|D|C|) \in \Delta^w \setminus \Delta^w_*$ are tolerated by $\Delta^s \cup \Delta^w_*$.*

The proof of propostion 1 follows immediately from definition 3 and 4. Moreover, choosing Δ^w_* as the least fixed point of Λ_Δ ensures that Δ^w_* is minimal in the sense that every $(|D|C|)$ outside of Δ^w_* is tolerated by the complement of Δ^w_*. The crucial feature of Δ^w_* is that it is the complement set of all weak conditionals that are tolerated by Δ and that it is monotonous in that sense, meaning if the set Δ gets reduced, also the set Δ^w_* reduces. We will make use of this property in the next section, when we present a consistency test for sets of standard and weak conditionals.

We can calculate Δ^w_* for finite sets of conditionals using algorithm 1. Note that, in steps 2 - 4 of algorithm 1, we assess $\Lambda_\Delta(\widetilde{\Delta}^w)$. The running time of algorithm 1 is determined by the SAT-test in line 2 and the size of $|\Delta^w| = m$. In the worst case, we get $\Delta^w_* = \Delta^w$ by adding a single weak conditional to $\Lambda_\Delta(\widetilde{\Delta}^w)$ each time we return to step 2 and we obtain $\mathcal{O}(m^2 t)$ where t represents the runtime of the SAT-test. We illustrate algorithm 1 by continuing example 1:

Algorithm 1: Calculation of Δ^w_* using the Λ_Δ operator.

Input: (Finite) set of conditionals $\Delta = \Delta^s \cup \Delta^w$.
Output: Set of limiting weak conditionals Δ^w_*

1 Initialize $\widetilde{\Delta}^w := \emptyset$ and $\Lambda_\Delta(\widetilde{\Delta}^w) := \emptyset$;
2 **foreach** $(|D|C|) \in \Delta^w$ **do**
3   **if** (1) is true **then** $\Lambda_\Delta(\widetilde{\Delta}^w) := \Lambda_\Delta(\widetilde{\Delta}^w) \cup (|D|C|)$;
4 **end**
5 **if** $\Lambda_\Delta(\widetilde{\Delta}^w) \neq \widetilde{\Delta}^w$ **then** $\widetilde{\Delta}^w := \Lambda_\Delta(\widetilde{\Delta}^w)$ and return to 2;
6 **else** $\Delta^w_* := \Lambda_\Delta(\widetilde{\Delta}^w)$;
7 Return Δ^w_*;

Example 2 (Continue example 1)**.** *We rewrite Δ from example 1 as $\Delta = \Delta^s \cup \Delta^w = \{(b|s), (v|s), (\bar{v}|b)\} \cup \{(|\bar{b}|\bar{s}|), (|s|v|)\}$ and determine Δ^w_* by applying algorithm 1 to Δ: We start with $(|\bar{b}|\bar{s}|)$. Since there exists $\omega \in \Omega$, so that $\omega \models \bar{s}\bar{b} \wedge (\bar{s} \vee b) \wedge (\bar{s} \vee v) \wedge (\bar{b} \vee \bar{v})$, we continue with $(|s|v|)$ without adding $(|\bar{b}|\bar{s}|)$ to $\Lambda_\Delta(\emptyset)$. For $(|s|v|)$,*

(1) *holds so* $\Lambda_\Delta(\emptyset) = \{(|s|v|)\}$ *and we have to check* (1) *again for* $(|\bar{b}|\bar{s}|)$. *Since* (1) *is still not fulfilled we obtain* $\Delta^w_* = \{(|s|v|)\}$.

4 System Z^w for standard and weak conditionals

In this section, we introduce a necessary and sufficient condition for consistency of sets $\Delta = \Delta^s \cup \Delta^w$ of conditionals, which enables us to define a ranking function κ, so that $\kappa \models \Delta$. Therefore, we extend System Z, a unifying schema for non-monotonic reasoning introduced by Pearl in (1990), for weak conditionals. First, we introduce a property of partitions of Δ that is crucial to decide whether Δ is consistent.

($*$) Let $\Delta = \Delta^s \cup \Delta^w$ be a set of standard and weak conditionals. An ordered partition $\Delta = (\Delta_0, \dots, \Delta_p) = (\Delta^s_0 \cup \Delta^w_0, \dots, \Delta^s_p \cup \Delta^w_p)$ with $\Delta_k \neq \emptyset$ $(1 \leqslant k \leqslant p)$ satisfies ($*$), if every $(Y/X) \in \Delta_k$ is tolerated by $\bigcup_{j \geqslant k} \Delta^s_j \cup \bigcup_{j \geqslant k+1} \Delta^w_j$.

($*$) extends the idea of tolerance partitions of Δ^s (Pearl 1990) that imply consistency for sets of (only) standard conditionals to sets of standard and weak conditionals $\Delta = \Delta^s \cup \Delta^w$ with respect to their weakened acceptance condition. Two weak conditionals from the same Δ^w_k, do not necessarily have to tolerate each other, since their falsification does not need to be strictly less plausible than their verification. We now present an algorithm, that determines a partition of $\Delta = (\Delta_0, \dots, \Delta_p)$ that satisfies ($*$). Each $\Delta_k \neq \emptyset$ $(0 \leqslant k \leqslant p)$, but Δ^w_k or Δ^s_k may be empty, otherwise the algorithm fails. In the course of this section, we will prove that this is equivalent to Δ being a consistent set of conditionals using a theorem that characterizes consistency based on subsets of Δ.

Algorithm 2: Partition algorithm for sets $\Delta = \Delta^s \cup \Delta^w$.

Input: Finite set of standard and weak conditionals $\Delta = \Delta^s \cup \Delta^w$.
Output: Partition $(\Delta_0, \dots, \Delta_p)$ satisfying ($*$)

1 Initialize $k := 0$;
2 **while** $\Delta^s \neq \emptyset$ **do**
3   Determine Δ^w_* for Δ via algorithm 1;
4     $\Delta^w_k := \Delta^w \setminus \Delta^w_*$;
5   **foreach** $(B|A) \in \Delta^s$ **do**
6   **if** $(B|A)$ is tolerated by $\Delta^s \cup \Delta^w_*$ **then** $(B|A) \in \Delta^s_k$;
7   **end**
8     $\Delta_k := \Delta^s_k \cup \Delta^w_k$;
9   **if** $\Delta_k = \emptyset$ **then** Return Failure;
10     $\Delta := (\Delta^s - \Delta^s_k) \cup (\Delta^w - \Delta^w_k)$
11     $k := k + 1$
12 **end**
13 **if** $\Delta^w \neq \emptyset$ **then** $k := k + 1$ and $\Delta_k = \Delta^w$;
14 $\Delta = (\Delta_0, \dots, \Delta_k)$

The running time of algorithm 2 is determined by the run time of algorithm 1 and the SAT-test in line 6 which depends on the size of $|\Delta^s| = n$. In the worst case, only one standard conditional is incorporated to the subsets Δ_k in each loop run in line 2, therefore we obtain $\mathcal{O}(n(m^2 t))$ where t represents the runtime of the SAT-test.

Theorem 1 clarifies the relation between Δ^w_* which is determined for every reduced set Δ from step 10 and the set $\bigcup_{j \geqslant k+1} \Delta^w_j$ for the subsets Δ_k of the partition:

Theorem 1. *Let* $\Delta = \Delta^s \cup \Delta^w$ *be a (consistent) set of conditionals and* $(\Delta_0, \dots, \Delta_p)$ *the partition determined using algorithm 2. Then* $(\Delta_0, \dots, \Delta_p)$ *satisfies* ($*$).

Proofsketch. Let $(Y/X) \in \Delta$ be a conditional. We start with $k = 0$ but the argumentation is the same for each k. Let $(|D_l|C_l|) \in \Delta_0$, then $(|D_l|C_l|) \notin \Delta^w_*$ (see step 4) and therefore, $\tilde{\omega} \in \Omega$ exists such that $\tilde{\omega} \models C_l D_l \wedge \bigwedge_{\substack{j \geqslant 0 \\ (B|A) \in \Delta^s_j}} (A \Rightarrow B) \wedge \bigwedge_{(|D|C|) \in \Delta^w_*} (C \Rightarrow D)$. Since Λ_Δ is a bounded operator, it holds that $\Delta^w_* \subseteq \Delta^w$. Hence, (see step 4 of algorithm 2) it holds that $\Delta^w_* = \Delta^w \setminus \Delta^w_0$, therefore it holds that $\tilde{\omega} \models C_l D_l \bigwedge_{\substack{j \geqslant 0 \\ (B|A) \in \Delta^s_j}} (A \Rightarrow B) \wedge \bigwedge_{\substack{j \geqslant 1 \\ (|D|C|) \in \Delta^w_j}} (C \Rightarrow D)$. So, for all $(|D_l|C_l|) \in \Delta_0$ ($*$) holds. For $(B_i|A_i) \in \Delta_0$, it holds that $(B_i|A_i)$ is tolerated by $\Delta^s \cup \Delta^w_*$ (see step 6). Therefore, we can follow the same argumentation as for weak conditionals. Hence, ($*$) holds for all $(B_i|A_i) \in \Delta_0$. Since Δ^w_* is determined for every (reduced) set of conditionals Δ according to line 10, the theorem follows for any $k \in \{0, \dots, p\}$. □

Note that, the set Δ^w_* is minimal in the sense that every conditional $(|D|C|) \in \Delta \setminus \Delta^w_*$ is tolerated by $\Delta^s \cup \Delta^w_*$, i.e. the set $\Delta^w_k = \Delta \setminus \Delta^w_*$ is the maximal set such that $(|D|C|) \in \Delta^w_k$ is tolerated by $\Delta^s \cup \Delta^w_*$.

Now, that we have seen that the partition $(\Delta_0, \dots, \Delta_p)$ satisfies ($*$), we need to assure that ($*$) guarantees that the set Δ is consistent. To prove that there exists a model of Δ, namely a ranking function κ so that $\kappa \models \Delta$ we extend System Z. By design of algorithm 1, the output of algorithm 2 is a partition $(\Delta_0, \dots, \Delta_p)$ that imposes an unambiguous and maximal ordering of the conditionals in Δ which satisfies ($*$). Note that we need to extend Pearl's approach for weak conditionals, since weak conditionals impose a less strict acceptance condition than standard conditionals on ranking functions (see definition 2). Therefore, we need to weaken their impact on the System Z ranking function. We call this ranking function *weak System Z* $\kappa^{z,w}$. To define $\kappa^{z,w}$, we first need to define a *Z^w-ordering of the conditionals in* Δ:

Definition 5 (Z^w-ordering of conditionals)**.** *Let* $\Delta = (\Delta_0, \dots, \Delta_p)$ *be the ordered partition of the conditionals in* Δ *which is computed by algorithm 2. We define the* Z^w *ranking of a conditional* $(Y/X) \in \Delta$ *as the number* $k \in \{1, \dots, s\}$ *of the subset* Δ_k *so that* $(Y/X) \in \Delta_k$*:*

$$Z^w((Y/X)) = k \textit{ iff } (Y/X) \in \Delta_k$$

Using the Z^w-ordering, we obtain the following $\kappa^{z,w}$, which we call *weak System Z*:

Definition 6. *Let* $\Delta = \Delta^s \cup \Delta^w$ *be a set of standard and weak conditionals and* $(\Delta_0, \dots, \Delta_p)$ *be the partition computed by algorithm 2 such that* ($*$) *is satisfied.* Weak System

Z *is obtained via Z^w-orderings $Z^w((Y/X))$ as defined in definition 5 by setting*

$$\kappa^{z,w}(\omega) = \begin{cases} 0, & \textit{if } \omega \textit{ does not falsify any conditional in } \Delta \\ \max\{\max_{1\leqslant i\leqslant n}\{Z^w((B_i|A_i))|\,\omega \models A_i\overline{B}_i\} \\ +1, \max_{1\leqslant l\leqslant m}\{Z^w(((|D_l|C_l|))|\,\omega \models C_l\overline{D}_l\}\}, \\ \quad \textit{otherwise}. \end{cases}.$$

The following theorem proves that $\kappa^{z,w}$ is a ranking function and that $\kappa^{z,w} \models \Delta$:

Theorem 2. *Let $\Delta = \Delta^s \cup \Delta^w$ be a set of conditionals and $(\Delta_0, \ldots, \Delta_p)$ be the partition computed by algorithm 2, then $\kappa^{z,w}$ is a ranking function and $\kappa^{z,w} \models \Delta$.*

Proofsketch. If algorithm 2 returns a partition then each $\Delta_k \neq \emptyset$ ($k \in \{1, \ldots, p\}$), in particular, $\Delta_0 \neq \emptyset$. If $(Y/X) \in \Delta_0$, then there exists a world $\omega \in \Omega$, s.t. ω does not falsify any rule from $\Delta^s \cup \bigcup_{j\geqslant 1} \Delta^w_j$, hence $\kappa^{z,w}(\omega) = 0$. So, $\kappa^{z,w}$ is an OCF. Now, we will show that $\kappa^{z,w} \models \Delta$: Let $(B_i|A_i) \in \Delta^s$, then $(B_i|A_i) \in \Delta_k$ for $k \in \{1, \ldots, p\}$. It holds $\kappa^{z,w}(A_iB_i) \leqslant k$, since there is $\omega \models A_iB_i \bigwedge_{\substack{j\geqslant k \\ (B|A)\in\Delta^s_j}} (A \Rightarrow B) \wedge \bigwedge_{\substack{j\geqslant k+1 \\ (|D|C|)\in\Delta^w_j}} (C \Rightarrow D)$. Furthermore, $\kappa^{z,w}(A_i\overline{B}_i) \geqslant k+1$ because $(B_i|A_i)$ is falsified. Hence, $\kappa^{z,w}(A_iB_i) < \kappa^{z,w}(A_i\overline{B}_i)$. Following the same argumentation for $(|D_l|C_l|) \in \Delta^w$ with $(|D_l|C_l|) \in \Delta_k$, we get $\kappa^{z,w}(C_lD_l) \leqslant k$ and $\kappa^{z,w}(C_l\overline{D}_l) \geqslant k$. Hence $\kappa^{z,w}(C_lD_l) \leqslant \kappa^{z,w}(C_l\overline{D}_l)$. □

Taking the partition from algorithm 2, i.e. the one with maximal partitioning sets Δ_k ($1 \leqslant k \leqslant p$), makes $\kappa^{z,w}$ unique. But note that this construction of a model of Δ would work for any partition of Δ satisfying $(*)$. Hence, we obtain the following corollary:

Corollary 1. *Let $\Delta = (\Delta_0, \ldots, \Delta_p)$ be a partition of $\Delta = \Delta^s \cup \Delta^w$ that satifies $(*)$, then Δ is consistent.*

The next theorem shows that the condition imposed on a partition of Δ by $(*)$ is also a necessary condition for the consistency of Δ:

Theorem 3. *Let $\Delta = \Delta^s \cup \Delta^w$ be a consistent set of conditionals. Then there exists a partition $(\Delta_0, \ldots, \Delta_p)$ that satisfies $(*)$.*

Proofsketch. Since Δ is consistent, there exists a κ with $\kappa \models \Delta$. Let $R_\Delta = \{\kappa(XY) | (Y/X) \in \Delta\} = \{\alpha_0, \ldots, \alpha_p\} \subset \mathbb{N}$ with $\alpha_0 < \ldots < \alpha_p$. We define a partition $(\Delta_0, \ldots, \Delta_p)$ of Δ as follows: $\Delta_k = \{(Y/X) \in \Delta \,|\, \kappa(XY) = \alpha_k\}$ for $k \in \{0, \ldots, p\}$. Because κ is consistent and $\kappa(XY)$ minimal, it holds for all $(B|A) \in \bigcup_{j\geqslant k} \Delta^s_j$ and $(|D|C|) \in \bigcup_{j\geqslant k+1} \Delta^w_j$, that $\alpha_k \leqslant \kappa(AB) < \kappa(A\overline{B})$ and $\alpha_k \leqslant \kappa(CD) \leqslant \kappa(C\overline{D})$. Therefore, for $(Y/X) \in \Delta_k$ and $\tilde{\omega} \in \Omega$ with $\kappa(\tilde{\omega}) = \kappa(XY) = \alpha_k$, it holds that $\tilde{\omega} \models XY \wedge \bigwedge_{\substack{j\geqslant k \\ (B|A)\in\Delta^s_j}} (A \Rightarrow B) \wedge \bigwedge_{\substack{j\geqslant k+1 \\ (|D|C|)\in\Delta^w_j}} (C \Rightarrow D)$, hence $(*)$ is satisfied. □

Note that the partition from the above proof is not necessarily the partition determined by algorithm 2, but rather a finer one, with more subsets $\Delta_k \subseteq \Delta$. Corollary 1 and theorem 3 together imply that a set Δ is consistent, if and only if there is a partition of $\Delta = (\Delta_0, \ldots, \Delta_p)$ that satisfies $(*)$. In the next theorem, we characterize consistency of sets of conditionals via the tolerance of subsets of Δ. This consistency condition will close the gap between the necessary and sufficient condition $(*)$ and algorithm 2. To prove our results the next lemma will be useful:

Lemma 1. *Let Δ be a set of conditionals and $(Y/X) \in \Delta$. If (Y/X) is not tolerated by $\tilde{\Delta} \subseteq \Delta$ then for all OCFs κ with $\kappa \models \Delta$, there is $(W/V) \in \tilde{\Delta}$ such that $\kappa(XY) \geqslant \kappa(V\overline{W})$.*

Proof. Since (Y/X) is not tolerated by $\tilde{\Delta}$, it holds that $XY \models \bigvee_{(W/V)\in\tilde{\Delta}} V\overline{W}$. Therefore, for any $\kappa \models \Delta$, we have $\kappa(XY) \geq \kappa(\bigvee_{(W/V)\in\tilde{\Delta}} V\overline{W}) = \min_{(W/V)\in\tilde{\Delta}}\{\kappa(V\overline{W})\}$. □

Theorem 4. *Let $\Delta = \Delta^s \cup \Delta^w$ be a set of conditionals. Δ is consistent iff for all subsets $\tilde{\Delta} \subseteq \Delta$ with $\tilde{\Delta}^s \neq \emptyset$, there exists a subset $\Delta'' \subseteq \tilde{\Delta}$ ($\Delta'' \neq \emptyset$), such that all $(Y/X) \in \Delta''$ are tolerated by $\tilde{\Delta} \setminus (\Delta'' \cap \tilde{\Delta}^w)$.*

Proof. "⇒": Let Δ be consistent, hence there exists κ with $\kappa \models \Delta$, and let $\tilde{\Delta} \subseteq \Delta$ with $\tilde{\Delta}^s \neq \emptyset$. If $\kappa(CD) < \kappa(C\overline{D})$ for all $(|D|C|) \in \tilde{\Delta}^w$, then $\{(Y/X) \mid (Y/X) \in \tilde{\Delta}\}$ is consistent and hence there exists $\Delta'' \subseteq \tilde{\Delta}, \Delta'' \neq \emptyset$, s.t. all $(Y/X) \in \Delta''$ are tolerated by $\tilde{\Delta}$. Otherwise $\Delta'_2 = \{(|D|C|) \in \tilde{\Delta}^w \mid \kappa(CD) = \kappa(C\overline{D})\} \neq \emptyset$. Let $\Delta' = \{(Y/X) \in \tilde{\Delta} \mid \kappa(XY) = \min_{(Y'/X')\in\tilde{\Delta}}\{\kappa(X'Y')\}\}$, $\Delta' \neq \emptyset$. Let $\Delta_1 = (\Delta' \cap \tilde{\Delta}^s) \subseteq \tilde{\Delta}^s$ and $\Delta_2 = (\Delta' \cap \Delta'_2) \subseteq \tilde{\Delta}^w$, consider $\Delta_1 \cup \Delta_2$.

Case $\Delta_1 = \emptyset = \Delta_2$: Then all $(Y/X) \in \Delta'$ are in $\tilde{\Delta}^w$ s.t. $\kappa(XY) < \kappa(X\overline{Y})$. Then all $(Y/X) \in \Delta'$ are tolerated by $\tilde{\Delta}$, because: Assume $(Y/X) \in \Delta'$ is not tolerated by $\tilde{\Delta}$, then $\kappa(XY) \geq \kappa(E\overline{F})$ for at least one $(F/E) \in \tilde{\Delta}$, hence $\kappa(XY) \geq \kappa(E\overline{F}) \geq \kappa(EF)$, therefore $(F/E) \in \Delta'$, so $\kappa(XY) > \kappa(EF)$, this contradicts $(Y/X) \in \Delta'$. Hence, all $(Y/X) \in \Delta'$ are also tolerated by $\tilde{\Delta} \setminus (\Delta' \cap \tilde{\Delta}^w)$.

Case $\Delta_1 = \emptyset, \Delta_2 = \Delta' \cap \Delta'_2 \neq \emptyset$: It holds for $(Y/X) \in \Delta'$ that $(Y/X) \in \tilde{\Delta}^w$, and there exists $(|D|C|) \in \tilde{\Delta}^w$ s.t. $\kappa(CD) = \min_{(|D'|C'|)\in\tilde{\Delta}}\{\kappa(X'Y')\}$ and $\kappa(CD) = \kappa(C\overline{D})$. Then all $(Y/X) \in \Delta' \cap \Delta'_2$ are tolerated by $\tilde{\Delta} \setminus (\Delta' \cap \Delta'_2)$ because: Assume there exists $(Y/X) \in \Delta' \cap \Delta'_2$ which is not tolerated by $\tilde{\Delta} \setminus (\Delta' \cap \Delta'_2)$ then there exists $(F/E) \in \tilde{\Delta} \setminus (\Delta' \cap \Delta'_2)$ s.t. $\kappa(XY) \geq \kappa(E\overline{F})$. For all $(F/E) \in \tilde{\Delta} \setminus (\Delta' \cap \Delta'_2)$, either $(F/E) = (F|E)$ and hence $\kappa(E\overline{F}) > \kappa(EF)$ holds, or $(F/E) = (|F|E|) \in \tilde{\Delta}^w \setminus (\Delta' \cap \Delta'_2)$ and hence also $\kappa(E\overline{F}) > \kappa(EF)$ holds. Therefore, in any case, $\kappa(XY) > \kappa(EF)$ which contradicts that $\kappa(XY)$ is minimal. So, $\Delta'' = \Delta' \cap \Delta'_2 \neq \emptyset, \Delta'' \subseteq \tilde{\Delta}^w$ and $\tilde{\Delta} \setminus (\Delta'' \cap \tilde{\Delta}^w) = \tilde{\Delta} \setminus \Delta'' = \tilde{\Delta} \setminus (\Delta' \cap \Delta'_2)$.

Case $\Delta_1 \neq \emptyset, \Delta_2 = \emptyset$: For all $(Y/X) \in \Delta'$, either $(Y/X) = (Y|X)$ with minimal $\kappa(XY) < \kappa(X\overline{Y})$, or $(Y/X) = (|Y|X|)$ with minimal $\kappa(XY) < \kappa(X\overline{Y})$. Let

$\Delta'' = \Delta' \cap \tilde{\Delta}^s = \Delta_1 \neq \emptyset$: Then all $(Y/X) \in \Delta''$ are tolerated by $\tilde{\Delta} \setminus (\Delta'' \cap \tilde{\Delta}^w) = \tilde{\Delta}$, because: Assume there exists $(Y/X) \in \Delta''$ s.t. (Y/X) is not tolerated by $\tilde{\Delta}$, then there exists $(F/E) \in \tilde{\Delta}$ s.t. $\kappa(XY) \geq \kappa(E\overline{F}) \geq \kappa(EF)$. Since $\kappa(XY)$ is minimal, it holds $\kappa(E\overline{F}) = \kappa(EF)$ and $\kappa(EF) = \kappa(XY)$. It follows $(F/E) \in \Delta' \cap \Delta'_2 = \Delta_2$ which contradicts $\Delta_2 = \emptyset$.

"$\Leftarrow$" Conversely, while $\Delta^s \neq \emptyset$, algorithm 2 determines a subset $\Delta_k = \Delta''$ of (reduced) Δ with $\Delta_k \neq \emptyset$. Hence, from theorem 1 it follows that Δ is consistent. □

The proof of theorem 4 shows that the consistency of a set Δ is highly dependent on the standard conditionals $\Delta^s \subseteq \Delta$. Yet, the addition of negative information in the form of weak conditionals imposes more restriction on the ordering of conditionals in a partition of Δ, therefore makes it harder to decide whether Δ is consistent or not. This corresponds to the observation, that a set Δ^w consisting only of weak conditionals is always consistent, since the uniform ranking function κ_u with $\kappa_u(\omega) = 0$ is always a model of Δ^w. From theorem 4 it follows that algorithm 2 is a consistency test for general sets $\Delta = \Delta^s \cup \Delta^w$.

Corollary 2. *Let $\Delta = \Delta^s \cup \Delta^w$ be a set of conditionals. Then Δ is consistent iff algorithm 2 determines a partition that satisfies $(*)$.*

Before we conclude, we will give an example of algorithm 2 and the corresponding $\kappa^{z,w}$. Therefore, we continue example 2:

Example 3 (Continue example 2)**.** *We apply algorithm 2 to the set $\Delta = \Delta^s \cup \Delta^w = \{(b|s), (v|s), (\bar{v}|b)\} \cup \{(|\bar{b}|\bar{s}|), (|s|v|)\}$ to determine whether it is consistent or not. From example 2 we know that $\Delta^w_* = \{(|s|v|)\}$ and therefore $\Delta^w_0 = \{(|\bar{b}|\bar{s}|)\}$. To determine Δ^s_0 we check which of the conditionals in Δ^s is tolerated by $\Delta^s \cup \Delta^w_*$ and obtain $\Delta^s_0 = \{(\bar{v}|b)\}$, since $b\bar{s}\bar{v} \models b\bar{v} \wedge (\bar{s} \vee b) \wedge (\bar{s} \vee v) \wedge (\bar{b} \vee \bar{v}) \wedge (\bar{v} \vee s)$. We start again in line 2 with the reduced $\Delta = \{(b|s), (v|s)\} \cup \{(|s|v|)\}$. Applying algorithm 1 we get that $\Delta^w_* = \emptyset$, so $\Delta^w_1 = \{(|s|v|)\}$. Since both conditionals in Δ^s are tolerated by $\Delta^s \cup \Delta^w_*$ we obtain $\Delta^s_1 = \{(b|s), (v|s)\}$ and the algorithm terminates. All in all, we get the following partition $\Delta = (\Delta_0, \Delta_1) = (\{(\bar{v}|b), (|\bar{b}|\bar{s}|)\}, \{(b|s), (v|s), (|s|v|)\})$, which satisfies $(*)$, and leads us to the following System Z^w ranking function $\kappa^{z,w}$ with $\kappa^{z,w}(\bar{s}b\bar{c}) = \kappa^{z,w}(\bar{s}\bar{b}\bar{c}) = 0$, $\kappa^{z,w}(sbv) = \kappa^{z,w}(\bar{s}bv) = \kappa^{z,w}(\bar{s}\bar{b}v) = 1$ and $\kappa^{z,w}(sb\bar{v}) = \kappa^{z,w}(s\bar{b}v) = \kappa^{z,w}(s\bar{b}\bar{v}) = 2$. It holds that $\kappa^{z,w} \models \Delta$ and therefore the belief base is consistent.*

5 Conclusion

Conditional belief bases with both positive and negative information play an important role in belief representation and non-monotonic reasoning. In this paper, we have extended the results of Booth and Paris from (Booth and Paris 1998) and provided a suitable extension of System Z for these mixed conditional belief bases. Via the close examination of the interplay between standard and weak conditionals, we defined the set of limiting weak conditionals as the fixed point of a monotonous and bounded operator Λ_Δ. Using Λ_Δ, we presented an algorithm that computes a partition of Δ. This partition imposes an ordering on the rules in Δ and serves a property $(*)$ which extends Pearl's idea of tolerance partition that imply consistency for mixed sets of conditionals. Also, $(*)$ enables us to define a non-trivial extension of System Z, called Weak System Z^w, for sets of standard and weak conditionals which induces preferences on models and plausible consequence relationships.

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